WIDE SUBCATEGORIES AND LATTICES OF TORSION CLASSES

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This talk is based on joint work [1] with Calvin Pfeifer (Bonn).

Let $\mathcal{A}$ be a fixed essentially small abelian length category. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$ is called a *torsion pair* if

$$\mathcal{F} = \mathcal{T}^{\perp} = \{X \in \mathcal{A} \mid \mathsf{Hom}_{\mathcal{A}}(\mathcal{T}, X) = 0\},$$
$$\mathcal{T} = {}^{\perp}\mathcal{F} = \{X \in \mathcal{A} \mid \mathsf{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0\}.$$

One can show that a full subcategory $\mathcal{T} \subset \mathcal{A}$ is completed to a torsion pair $(\mathcal{T}, \mathcal{T}^{\perp})$ if and only if $\mathcal{T}$ is closed under extensions and factor objects. We call such subcategories $\mathcal{T}$ *torsion classes* in $\mathcal{A}$, and then, the set $\mathsf{tors}\,\mathcal{A}$ partially ordered by inclusion is a complete lattice, that is, meets and joins are well-defined for all subsets of $\mathsf{tors}\,\mathcal{A}$.

For two torsion classes $\mathcal{U} \subset \mathcal{T}$ in $\mathcal{A}$, we can consider the interval $[\mathcal{U}, \mathcal{T}]$ in $\mathsf{tors}\,\mathcal{A}$ and a full subcategory $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$. The full subcategory $\mathcal{W}$ indicates the difference of the torsion classes $\mathcal{U} \subset \mathcal{T}$; more precisely,

$$\mathcal{T} = \mathcal{U} * \mathcal{W} := \{X \in \mathcal{A} \mid \text{there exists } 0 \to U \to X \to W \to 0 \text{ with } U \in \mathcal{U} \text{ and } W \in \mathcal{W}\}.$$

We call $[\mathcal{U}, \mathcal{T}]$ a *wide interval* if $\mathcal{W}$ is a *wide subcategory*, that is, $\mathcal{W}$ is closed under taking kernels, cokernels, and extensions. In this case, $\mathcal{W}$ is an abelian subcategory of $\mathcal{A}$ closed under extensions, so we have another complete lattice $\mathsf{tors}\,\mathcal{W}$.

A typical example of wide intervals is given by *τ-tilting reduction* established by Jasso [3] and Demonet–Iyama–Reading–Reiten–Thomas [2]; namely, let A be a finite-dimensional algebra over a field K, then a τ-rigid pair (N, Q) in the module category $\mathsf{mod}\,A$ gives a wide interval $[\mathsf{Fac}\,N, {}^{\perp}(\tau N) \cap Q^{\perp}]$. They showed that the wide interval $[\mathsf{Fac}\,N, {}^{\perp}(\tau N) \cap Q^{\perp}]$ is isomorphic to $\mathsf{tors}\,C_{N,Q}$ as a complete lattice, where $C_{N,Q}$ is a certain finite-dimensional K-algebra constructed from the τ-rigid pair (N, Q).

In our study, we were able to extend their result to all wide intervals.

Theorem 1. *Let $[\mathcal{U}, \mathcal{T}]$ be a wide interval in $\mathsf{tors}\,\mathcal{A}$ and $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$. Then we have mutually inverse isomorphisms of complete lattices*

$$\Phi \colon [\mathcal{U}, \mathcal{T}] \to \mathsf{tors}\,\mathcal{W}, \quad \Psi \colon \mathsf{tors}\,\mathcal{W} \to [\mathcal{U}, \mathcal{T}]$$

given by $\Phi(\mathcal{V}) := \mathcal{U}^{\perp} \cap \mathcal{V}$ and $\Psi(\mathcal{X}) := \mathcal{U} * \mathcal{X}$ for any $\mathcal{V} \in [\mathcal{U}, \mathcal{T}]$ and any $\mathcal{X} \in \mathsf{tors}\,\mathcal{W}$.

In this talk, I would like to explain the detail of the theorem above. If time permits, I will give several characterizations of wide intervals obtained in our study.

References

1. S. Asai, C. Pfeifer, *Wide subcategories and lattices of torsion classes*, arXiv:1905.01148.
2. L. Demonet, O. Iyama, N. Reading, I. Reiten, H. Thomas, *Lattice theory of torsion classes*, arXiv:1711.01785.
3. G. Jasso, *Reduction of τ-tilting modules and torsion pairs*. Int. Math. Res. Not. IMRN 2015, no. 16, 7190–7237.

2010 *Mathematics Subject Classification.* 06A07, 18E40.