Ulrich ideals in hypersurfaces

Ryotaro Isobe
Chiba University
Email: r.isobe.math@gmail.com

The purpose of this talk is to investigate the structure and ubiquity of Ulrich ideals in a hypersurface ring.

In a Cohen-Macaulay local ring $(R, \mathfrak{m})$, an $\mathfrak{m}$-primary ideal I is called an Ulrich ideal in R if there exists a parameter ideal Q of R such that $I \supsetneq Q$, $I^2 = QI$, and I/I^2 is R/I-free. The notion of Ulrich ideal/module dates back to the work [3] in 2014, where S. Goto, K. Ozeki, R. Takahashi, K.-i. Watanabe, and K.-i. Yoshida introduced the notion, generalizing that of maximally generated maximal Cohen-Macaulay modules ([1]), and started the basic theory. The maximal ideal of a Cohen-Macaulay local ring with minimal multiplicity is a typical example of Ulrich ideals, and the higher syzygy modules of Ulrich ideals are Ulrich modules. In [3, 4], all Ulrich ideals of Gorenstein local rings of finite CM-representation type with dimension at most 2 are determined by means of the classification in the representation theory.

Nevertheless, even for the case of hypersurface rings, there seems known only scattered results which give a complete list of Ulrich ideals, except the case of finite CM-representation type and the case of several numerical semigroup rings. Therefore, in this talk, we focus our attention on a hypersurface ring which is not necessarily finite CM-representation type.

In what follows, unless otherwise specified, let $(S, \mathfrak{n})$ be a Cohen-Macaulay local ring with $\dim S = d+1$ $(d \geq 1)$, and $f \in \mathfrak{n}$ a non-zero divisor on S. We set $R = S/(f)$. For each $a \in S$, let $\overline{a}$ denote the image of a in R. We denote by $\mathcal{X}_R$ the set of Ulrich ideals in R. We then have the following, which characterizes Ulrich ideals in a hypersurface ring.

Theorem 1. *Suppose that $(S, \mathfrak{n})$ is a regular local ring with $\dim S = d+1$ $(d \geq 1)$ and $0 \neq f \in \mathfrak{n}$. Set $R = S/(f)$. Then we have*

$$\mathcal{X}_R = \left\{ (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \;\middle|\; \begin{array}{l} a_1, a_2, \ldots, a_d, b \in \mathfrak{n} \textit{ be a system of parameters of } S, \\ \textit{and there exist } x_1, x_2, \ldots, x_d \in (a_1, a_2, \cdots, a_d, b) \textit{ and } \varepsilon \in U(S) \\ \textit{such that } b^2 + \displaystyle\sum_{i=0}^{d} a_i x_i = \varepsilon f. \end{array} \right\},$$

where $U(S)$ denotes the set of unit elements of S.

Let $a_1, \ldots, a_d, b \in \mathfrak{n}$ be a system of parameters of S, so that $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$ with $x_1, \ldots, x_d \in (a_1, \cdots, a_d, b)$ and $\varepsilon \in U(S)$. Then $I = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d}, \overline{b}) \in \mathcal{X}_R$, with a reduction $Q = (\overline{a_1}, \overline{a_2}, \cdots, \overline{a_d})$ by Theorem 1. By [3, Corollary 7.2], in the exact sequence $0 \to Q \xrightarrow{\iota} I \to R/I \to 0$, the free resolution of I induced from minimal free resolutions of Q and R/I is also minimal. We construct this resolution, by using the relation $b^2 + \sum_{i=1}^{d} a_i x_i = \varepsilon f$. We set

$$K = \mathrm{K}_\bullet(a_1, \ldots, a_d; S) = (K_\bullet, \partial_\bullet^K) \text{ and } L = \mathrm{K}_\bullet(x_1, \ldots, x_d; S) = (K_\bullet, \partial_\bullet^L)$$

are Koszul complexes of S generated by $a_1, \ldots, a_d$ and $x_1, \ldots, x_d$. We define $G = (G_\bullet, \partial_\bullet)$ by $G_0 = K_0$, $G_i = K_i \oplus G_{i-1} = S^{\oplus \sum_{j=0}^{i} \binom{d}{j}}$ for $i \geq 1$, and

$$\partial_1 = \left[\begin{array}{c|c} \partial_1^K & b \end{array}\right], \partial_2 = \left[\begin{array}{c|c} \partial_2^K & -bE_d \mid {}^t\partial_1^L \\ \hline O & \partial_1 \end{array}\right], \text{ and}$$

$$\partial_i = \left[\begin{array}{c|c} \partial_i^K & (-1)^{i-1}bE_{\binom{d}{i-1}} \mid {}^t\partial_{i-1}^L \mid O \\ \hline O & \partial_{i-1} \end{array}\right] \text{ for } i \geq 3.$$

We notice that $\partial_i = \partial_{d+1}$ for any $i \geq d+1$. Set $F = (F_\bullet, \overline{\partial_\bullet}) = (G_\bullet \otimes R, \partial_\bullet \otimes R)$. We then have the following.

Theorem 2. *$F : \cdots \to F_i \xrightarrow{\overline{\partial_i}} F_{i-1} \to \cdots \to F_1 \xrightarrow{\overline{\partial_1}} F_0 = R \xrightarrow{\varepsilon} R/I \to 0$ is a minimal free resolution of R/I.*

As a consequence, we get a matrix factorization of d-th syzygy module of R/I, which is an Ulrich module with respect to I (see [3, Definition 1.2]).

Corollary 3. *Let $M = \operatorname{Im} \overline{\partial_d}$. Then $0 \to G_{d+2} \xrightarrow{\partial_{d+1}} G_{d+1} \xrightarrow{\tau} M \to 0$ is exact as S-modules and $\partial_{d+1}^2 = gE_{2^d}$, where $\tau : G_{d+1} \xrightarrow{\varepsilon} F_{d+1} \xrightarrow{\overline{\partial_d}} M$. Therefore ∂_{d+1} gives a matrix factorization of M.*

References

1. J. P. Brennan, J. Herzog, and B. Ulrich, Maximally generated maximal Cohen-Macaulay modules, *Math. Scand.*, **61** (1987), no. 2, 181–203.
2. S. Goto, Ulrich ideals in $k[[X,Y]]/(Y^3)$ and $k[[X,Y]]/(X^2Y)$, *The Proceedings of the 40-th Symposium on Commutative Algebra*, in preparation.
3. S. Goto, K. Ozeki, R. Takahashi, K.-i. Yoshida, and K.-i. Watanabe, Ulrich ideals and modules, *Math. Proc. Camb. Phil. Soc.*, **156** (2014), 137–166.
4. S. Goto, K. Ozeki, R. Takahashi, K.-i. Yoshida, and K.-i. Watanabe, Ulrich ideals and modules over two-dimensional rational singularities, *Nagoya Math. J.*, **221** (2016), 69–110.
5. R. Isobe, The structure of Ulrich ideals in hypersurfaces, arXiv:1905. 02048.

2010 *Mathematics Subject Classification.* 13D02, 13H10, 13H15.