An application of Hochschild cohomology to the moduli of subalgebras of the full matrix ring II

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Definition 1. We say that a subsheaf $\mathcal{A}$ of $\mathcal{O}_X$-algebras of $\mathrm{M}_n(\mathcal{O}_X)$ is a *mold* of degree n on a scheme X if $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ is a locally free sheaf. We denote by $\mathrm{rank}\mathcal{A}$ the rank of $\mathcal{A}$ as a locally free sheaf.

Proposition 2. *The following contravariant functor is representable by a closed subscheme of the Grassmann scheme* $\mathrm{Grass}(d, n^2)$*:*

$$\begin{array}{rccl} \mathrm{Mold}_{n,d} & : & (\mathbf{Sch})^{op} & \rightarrow & (\mathbf{Sets}) \\ & & X & \mapsto & \left\{ \mathcal{A} \,\middle|\, \mathcal{A} \textit{ is a mold of degree } n \textit{ on } X \textit{ with } \mathrm{rank}\mathcal{A} = d \right\}. \end{array}$$

The following theorem is one of the main results.

Theorem 3 ([1])**.** *Let S be a locally noetherian scheme. For a rank d mold $\mathcal{A}$ of degree n on S, denote by $\tau_{\mathcal{A}} : S \to \mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$ the morphism induced by $\mathcal{A}$. Set $\mathcal{A}(x) := \mathcal{A} \otimes_{\mathcal{O}_S} k(x) \subseteq \mathrm{M}_n(k(x))$, where $k(x)$ is the residue field of a point $x \in S$. Put $\mathrm{PGL}_{n,S} := \mathrm{PGL}_n \otimes_{\mathbb{Z}} S$. Let us define the S-morphism $\phi_{\mathcal{A}} : \mathrm{PGL}_{n,S} \to \mathrm{Mold}_{n,d} \otimes_{\mathbb{Z}} S$ by $P \mapsto P\mathcal{A}P^{-1}$. Then $\phi_{\mathcal{A}}$ is smooth if and only if $H^1(\mathcal{A}(x), \mathrm{M}_n(k(x))/\mathcal{A}(x)) = 0$ for each $x \in S$.*

Let k be an algebraically closed field. There are 26 types of k-subalgebras of $\mathrm{M}_3(k)$ up to inner automorphisms of $\mathrm{M}_3(k)$. For all types of k-subalgebras A of $\mathrm{M}_3(k)$, we have calculated Hochschild cohomology $H^i(A, \mathrm{M}_n(k)/A)$. We introduce several results not only for an algebraically closed field k but also for any commutative ring R.

Theorem 4 ([1])**.** *Set* $\mathrm{N}_3(R) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d \in R \right\} \subset \mathrm{M}_3(R)$ *for a commutative ring R. Then*

$$H^i(\mathrm{N}_3(R), \mathrm{M}_3(R)/\mathrm{N}_3(R)) = \begin{cases} R^2 & (i = 0) \\ R^{i+1} & (i > 0). \end{cases}$$

Theorem 5 ([1])**.** *Set* $\mathrm{S}_4(R) := \left\{ \begin{pmatrix} a & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix} \middle| \, a, b, c \in R \right\} \subset \mathrm{M}_3(R)$ *for a commutative ring R. Then*

$$H^i(\mathrm{S}_4(R), \mathrm{M}_3(R)/\mathrm{S}_4(R)) = \begin{cases} R^4 & (i = 0) \\ R^{3 \cdot 2^i} & (i > 0). \end{cases}$$

References

1. K. Nakamoto and T. Torii, *Applications of Hochschild cohomology to the moduli of subalgebras of the full matrix ring*, in preparation.

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