Pure derived categories and weak balanced big Cohen-Macaulay modules

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Let R be a commutative noetherian ring of finite Krull dimension. In the first half of this talk, we give a new approach to reach the pure derived category of flat R-modules. Motivated by Neeman [5], Murfet and Salarian [3] defined the pure deived category as the Verdier quotient $\mathsf{K}(\mathsf{Flat}\, R)/\mathsf{K}_{\mathsf{pac}}(\mathsf{Flat}\, R)$ of the homotopy category of complexes of flat R-modules by the subcategory of pure acyclic complexes. There is a general theory due to Gillespie [1] that yields complete cotorsion pairs in the level of complexes, and it is possible to deduce from his work that the pure derived category is triangulated equivalent to the homotopy category $\mathsf{K}(\mathsf{FlCot}\, R)$ of complexes of flat cotorsion modules, where we say that an R-module M is cotorsion if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat R-module F.

On the other hand, our main tool is a Čech complex of functors introduced in the previous work [4] with Yoji Yoshino. The Čech complex is constructed from localizations and completions with respect to prime ideals, and it yields a triangulated functor $\mathsf{K}(\mathsf{Flat}\, R) \to \mathsf{K}(\mathsf{FlCot}\, R)$. We prove that this functor is a left adjoint to the inclusion functor $\mathsf{K}(\mathsf{FlCot}\, R) \to \mathsf{K}(\mathsf{Flat}\, R)$, and this adjoint pair naturally induces the triangulated equivalence $\mathsf{K}(\mathsf{Flat}\, R)/\mathsf{K}_{\mathsf{pac}}(\mathsf{Flat}\, R) \xrightarrow{\cong} \mathsf{K}(\mathsf{FlCot}\, R)$. Moreover, using this fact, we concretely illustrate correspondence between different stable categories.

In the second half of this talk, we provide a reasonable framework to study an infinite version of Cohen-Macaulay representation theory. Following Holm [2], we say that an R-module M is *weak balanced big Cohen-Macaulay* if any system of parameters of the maximal ideal $\mathfrak{m}$ is a weak regular sequence on M, where $M/\mathfrak{m}M$ can be zero. If R is a Gorenstein local ring, then the subcategory $\mathsf{K}_{\mathsf{ac}}(\mathsf{FlCot}\, R)$ of acyclic complexes can be identified with the stable category of weak balanced big Cohen-Macaulay cotorsion modules modulo flat cotorsion modules. We explain that this stable category is suitable to develop Puninski's work [6].

References

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