Is Ware's problem true or not ?

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R. Ware gave the following problem in his paper: *Endomorphism rings of projective modules*, Trans. Amer. Math. Soc. **155** (1971), 233-256.

Problem: If a projective right R-module P has unique maximal submodule L, then L is the largest maximal submodule of P.

In the paper, A. Facchini, D. Herbera, I. Sakhajev *Finitely Generated Flat Modules and a Characterization of Semiperfect Rings*, Comm. in Algebra, Vol.**31** No.9(2003), 4195–214 asserts this problem is negative by showing the following properties:

Let ${}_RU$ be auniserial R-module and $S = \mathrm{End}_R(U)$ an endomorphism ring of ${}_RU$. Then he following conditions are equivalent.

(1) *U_S is not quasi-small.*

(2) *U_S is countable generated and a simple left R-module ${}_RR/K$ is flat and $\sum_{f\in K} f(U_S) = U_S$. Here $K = \{f \in S | f$ is not epimorphism$\}$.*

In this case, ${}_RK$ is an infinitely generated projective module with unique maximal submodule.

Here, U_S is called quasi-small if $U \cong T$ for a direct summand T of $\bigoplus_{i\in\Gamma} M_i$, then there is a finite subset $\Delta \subset \Gamma$ such that $T \subset \bigoplus_{i\in\Delta} M_i$. We emark T is a direct summand of $\bigoplus_{i\in\Delta} M_i$.

In this talk, we give some interesting example:

Example: *Let F be a field Z a commutative F-algebra with bases $\{v_x\ |0 < x \leq 1\}$ with the multiplication $v_x \cdot v_y = v_{xy}$*

which seems to be a counter example of the above properties.

Also we report Ware's problem is true by using Nakayama-Azumaya Lemma for projective modules.

Furter, we invesitage sturctures of a module with unique maximal submodule.

One structure theorem is:

Theorem: *Let R be a ring and M a right R-module with unique maximal submodule L. then M is indecomposable or $M = M_1 \oplus M_2$ such that*

M_1 has unique maximal submodule and M_2 does not have any maximal submodules.

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