Action functor formalism

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Given a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, 1)$, we denote its Drinfeld center by $\mathcal{Z}(\mathcal{C})$. If the forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$ admits a right adjoint, say R, then the adjoint object of $\mathcal{C}$ is defined by $A_{\mathcal{C}} := UR(1)$. Our main concern is the case where $\mathcal{C}$ is a finite tensor category in the sense of Etingof-Ostrik [1]. Some fundamental results on finite-dimensional Hopf algebras have been extended to the setting of finite tensor categories by using the adjoint object and the adjunction $U \dashv R$ [3, 4]. Here a naive question arises: Why is the adjoint object useful for this kind of problems? As the adjoint object is defined in terms of the tensor product of $\mathcal{C}$, there is no obvious reason why it relates to somewhat ring-theoretic or representation-theoretic problems.

In this talk, I introduce an abstract framework connecting the adjoint object and several ring-theoretic notions and review how results on Hopf algebras are extended to the setting of finite tensor categories. Let $\mathcal{C}$ be a finite tensor category. A key ingredient is the 'action' functor $\rho : \mathcal{C} \to \mathrm{Rex}(\mathcal{C})$ defined by $\rho(X) = X \otimes (-)$, where $\mathrm{Rex}(\mathcal{C})$ is the category of right exact linear endofunctors on $\mathcal{C}$. It turns out that ρ has a right adjoint, say ρ^{ra}, and the adjoint object $A_{\mathcal{C}}$ is isomorphic to $\rho^{\mathrm{ra}}(\mathrm{id}_{\mathcal{C}})$. If we pick an arbitrary algebra L such that $\mathcal{C} \approx L$-mod, then $\mathrm{Rex}(\mathcal{C}) \approx L$-bimod. Some ring-theoretic notions can be formulated in terms of the category of bimodules. If a ring-theoretic notion which we aim to investigate has such a description, then one can transport it to the category $\mathcal{C}$ through the equivalence L-bimod $\approx \mathrm{Rex}(\mathcal{C})$ and the functor $\rho^{\mathrm{ra}} : \mathrm{Rex}(\mathcal{C}) \to \mathcal{C}$. This allows us to discuss relations between the notion and the adjoint object.

As explained in [5], this formalism has a lot of applications. For example, $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(1, A_{\mathcal{C}})$ is shown to be isomorphic to the Hochschild cohomology $\mathrm{HH}^{\bullet}(L)$. Noteworthy, this result extends the $\mathrm{SL}_2(\mathbb{Z})$-action on the Hochschild cohomology of a ribbon factorisable Hopf algebra to the setting of non-semisimple modular tensor categories. Under the assumption that the double dual functor on $\mathcal{C}$ is isomorphic to the identity functor, $\mathrm{Ext}^{\bullet}_{\mathcal{C}}(A_{\mathcal{C}}, 1)$ is shown to be dual to Hochschild homology $\mathrm{HH}_{\bullet}(L)$ by a similar argument and an abstract treatment of the Nakayama functor established in [2]. Thus, under the same assumption, $\mathrm{Hom}_{\mathcal{C}}(A_{\mathcal{C}}, 1)$ is isomorphic to the space of symmetric linear forms on L. I will show further applications of this kind of techniques. If time permits, I will talk about a generalization to modules over a finite tensor category.

References

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