Auslander-Bridger theory for projective complexes over commutative Noetherian rings

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Let R be a commutative Noetherian ring and let $\mathcal{K}(R)$ be the homotopy category of all complexes of finitely generated projective modules over R. For any $X \in \mathcal{K}(R)$ the R-dual complex $X^* = \mathrm{Hom}_R(X, R)$ is defined and the operation $(-)^*$ gives the duality on $\mathcal{K}(R)$. The main theorem of this talk is the following:

Main Theorem [2] *Let $X \in \mathcal{K}(R)$ and assume that R is a generically Gorenstein ring. Then, X is acyclic if and only if X^* is acyclic.*

Recall that R is called a generically Gorenstein ring if the total ring of quotients is Gorenstein. This theorem includes the Tachikawa conjecture and the dependence of totally reflexivity conditions for modules over a generically Gorenstein ring.

To prove this theorem we need to develop and establish the Auslander-Bridger type theory for $\mathcal{K}(R)$. Precisely speaking, we have a natural mapping $\rho^i_{X,R} : H^{-i}(X^*) \to H^i(X)^*$ for $X \in \mathcal{K}(R)$ and $i \in \mathbb{Z}$. We say that a complex $X \in \mathcal{K}(R)$ is ***torsion-free** (resp. ***reflexive**) if $\rho^i_{X,R}$ are injective (resp. bijective) mappings for all $i \in \mathbb{Z}$. Let $\mathrm{Add}(R)$ be the additive full subcategory of $\mathcal{K}(R)$ consisting of all split complexes. We can show that $\mathrm{Add}(R)$ is functorially finite in $\mathcal{K}(R)$ and hence every complex in $\mathcal{K}(R)$ is resolved by complexes in $\mathrm{Add}(R)$. Define $\underline{\mathcal{K}(R)}$ to be the factor category $\mathcal{K}(R)/\mathrm{Add}(R)$. Then we are able to define the syzygy functor Ω and the cosyzygy functor Ω^{-1} on $\underline{\mathcal{K}(R)}$, and as a result we have an adjoint pair (Ω^{-1}, Ω) of functors. Then we can show that X is *torsion-free iff $X \cong \Omega^{-1}\Omega X$ in $\underline{\mathcal{K}(R)}$. And under the assumption that R is generically Gorenstein, X is *reflexive iff $X \cong \Omega^{-2}\Omega^2 X$ in $\underline{\mathcal{K}(R)}$.

There is a triangles of the form

$$\Delta^{(n,0)}(X) \to \Omega^{-n}\Omega^n(X) \to X \to \Delta^{(n,0)}(X)[1],$$

for $X \in \mathcal{K}(R)$ and $n > 0$, where $\Delta^{(n,0)}(X)$ has a finite $\mathrm{Add}(R)$-resolution of length at most $n-1$. This is one of the key theorems in order to prove Main Theorem. The second key observation is that any syzygy complex $\Omega^r X$ $(\forall r > 0)$ is *torsion-free if $H(X^*) = 0$.

References

1. M. Auslander, M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94 American Mathematical Society, Providence, R.I. (1969), 146 pp.
2. Yuji Yoshino, *Homotopy categories of unbounded complexes of projective modules*, arXiv:1805.05705v3.