On cyclotomic quiver Hecke algebras of affine type

Susumu Ariki

Osaka University

The Eighth
China - Japan - Korea International Symposium
on Ring Theory

Nagoya University

August 26-31 (2019)

Objects to study

We know the symmetric group S_n. The group algebra is generated by Coxeter generators $s_1, \dots, s_{n-1}$ and subject to the Coxeter relations.

We may introduce a parameter q to deform the group algebra to the Hecke algebra. Through the development in the past decades, they have been generalized to cyclotomic Hecke and the cyclotomic quiver Hecke algebras associated with Lie theoretic data.

Those algebras are the objects we want to study.

The key to introduce the latter algebras was the discovery of the Khovanov-Lauda(-Rouquier) generators.

One consequence by Brundan and Kleshchev:

The group algebra of the symmetric group is a graded algebra.

In this talk, I shall explain how Fock spaces from physics plays a role in the study of cyclotomic quiver Hecke algebras of affine Lie type.

Soliton theory

Let us begin by the paper

Transformation Groups for Soliton Equations
– Euclidean Lie Algebras and
Reduction of the KP Hierarchy –

by Etsuro Date, Michio Jimbo, Masaki Kashiwara and Tetsuji Miwa, which was published in Publ. RIMS, Kyoto Univ. **18** (1982), 1077–1110.

There, they write the Kadomtsev-Petviashvli equations (the KP equations for short) and their reductions in the Hirota bilinear form. For example, the famous KdV equation has the Hirota form $(D_1^4 - 4D_1D_3)\tau\cdot\tau = 0$, where $P(D_1, D_2, \dots)\tau\cdot\tau$ is defined by

$$P(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots)\tau(x_1+y_1, x_2+y_2, \dots)\tau(x_1-y_1, x_2-y_2, \dots)|_{y_1=y_2=\dots=0}.$$

Soliton theory (cont'd)

Those nonlinear differential equations admit the infinitesimal symmetry $\mathfrak{gl}(\infty)$, i.e. the central extension of

$$\left\{ \sum_{i,j\in\mathbb{Z}} a_{ij}E_{ij} \mid a_{ij} = 0, \text{if } |i-j| \text{ is sufficiently large.} \right\},$$

or its reduction (Chevalley generators are infinite sums of E_{ij}'s), namely

the set of τ-functions form the orbit through $|vac\rangle$:

$\tau(x) = \langle vac| \exp H(x)|L\rangle$, *where L runs through the orbit,*

in the Fock representation of the infinite dimensional Lie algebra.

The similar result holds for the BKP equations, where the infinitesimal symmetry is $\mathfrak{go}(\infty)$.

Soliton theory (cont'd)

The reduction mentioned above is a simple procedure to obtain Chevalley generators of the affine Lie algebras as infinite sums of the generators of $\mathfrak{gl}(\infty)$ or $\mathfrak{go}(\infty)$.

In this setup, the affine Lie algebras $A_\ell^{(1)}$ appear as the reduction of the KP hierarchy, $A_{2\ell}^{(2)}$ and $D_{\ell+1}^{(2)}$ appear as the reduction of the BKP hierarchy.

The Fock space for the KP or the BKP is based on charged fermions or neutral fermions, respectively. We may rewrite those Fock spaces in terms of partitions/shifted partitions, which form a basis of the Fock space.

The nodes of partitions/shifted partitions are given residue $0, 1, \ldots, \ell$ and the action of the Chevalley generator f_i on each of the partitions/shifted partitions adds one node of residue i.

Affine Dynkin diagrams arising from the soliton theory

We consider the affine Dynkin diagrams

$$A_{2\ell}^{(2)}(=\widetilde{BC}_\ell),\ \ D_{\ell+1}^{(2)}(=\widetilde{B}_\ell),\ \ A_\ell^{(1)}(=\widetilde{A}_\ell),\ \ C_\ell^{(1)}(=\widetilde{C}_\ell)$$

in this talk. (All of them have $\ell+1$ vertices.)

We will introduce cyclotomic quiver Hecke algebras (aka cyclotomic KLR algebras), and we will use those Fock spaces to obtain dimension formulas for the algebras.

Remark 2.1

In their paper, $A_{2\ell-1}^{(2)}(=\widetilde{CD}_\ell)$ and $D_\ell^{(1)}$ appear as the reductions from the two component BKP. But we do not use them here.

Combinatorial Fock spaces : $A_{2\ell}^{(2)}$

$$A_{2\ell}^{(2)} = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

When $\ell = 1$, the Cartan matrix of type $A_2^{(2)}$ is $\left(\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right)$.

Combinatorial Fock spaces : $A^{(2)}_{2\ell}$ (cont'd)

The Fock space is the vector space whose basis is the set of shifted partitions. We color the nodes of shifted partitions with the residue pattern which repeats $01\cdots\ell\cdots10$ *in each row.*

For example, if $\ell = 2$ then

0	1	2	1	0	0
	0	1			

Combinatorial Fock spaces : $D^{(2)}_{\ell+1}$

$$D^{(2)}_{\ell+1} = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -2 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

When $\ell = 1$, the Cartan matrix of type $D^{(2)}_2$ is defined to be $A^{(1)}_1$.

Combinatorial Fock spaces : $D^{(2)}_{\ell+1}$ (cont'd)

The Fock space is the same as $A^{(2)}_{2\ell}$ but the residue pattern is different. It repeats $01\cdots\ell\ell\cdots10$ in each row.

For example, if $\ell = 2$ then

0	1	2	2	1	0
	0	1			

Combinatorial Fock spaces : $A_\ell^{(1)}$

$$A_\ell^{(1)} = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

When $\ell = 1$, the Cartan matrix of type $A_1^{(1)}$ is $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$.

Combinatorial Fock spaces : $A_\ell^{(1)}$ (cont'd)

The Fock space is the vector space whose basis is the set of partitions. The residue pattern involves $s \in \mathbb{Z}/(\ell+1)\mathbb{Z}$:

> *the cell on the r^{th} row and the c^{th} column of a partition has the residue $s - r + c$ modulo $\ell + 1$.*

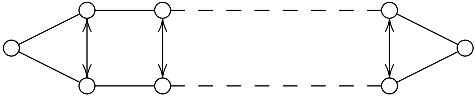
We denote the Fock space with the residue pattern by $\mathcal{F}_s$.

For example, if $\ell = 2$ and $s = 1$ then

1	2	0	1	2	0
0	1	2			

Folding $A^{(1)}_{2\ell-1}$ to $C^{(1)}_{\ell}$

By the folding procedure, we obtain $C^{(1)}_{\ell}$ from $A^{(1)}_{2\ell-1}$.



$$C^{(1)}_{\ell} = (a_{ij})_{i,j\in I} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -2 \\ 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

Combinatorial Fock spaces : $C_\ell^{(1)}$

Recall that the residue pattern of the Fock space $\mathcal{F}_{s=0}$ for $A_\ell^{(1)}$ is

$$\begin{array}{ccccccc} 0 & 1 & \cdots & \ell & 0 & 1 & \cdots \\ \ell & 0 & 1 & \cdots & \ell & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{array}$$

namely, we repeat $01\cdots\ell$ on the rim. We slide the residue sequence on the rim to obtain $\mathcal{F}_s$, where s sits on the corner instead of 0.

> *The Fock space for $C_\ell^{(1)}$ is the same as $A_\ell^{(1)}$, but the residue sequence on the rim repeats $01\cdots\ell\cdots21$. We denote the Fock space by $\mathcal{F}_s$ again.*

Integrable highest weight modules

- In all the cases $A = A^{(2)}_{2\ell}, D^{(2)}_{\ell+1}, A^{(1)}_{\ell}, C^{(1)}_{\ell}$, the residue pattern defines the action of the Kac-Moody Lie algebra $\mathfrak{g}(A)$.
- In the Fock spaces, the vacuum vector generates the highest weight module $V(\Lambda_0)$ of the corresponding affine Lie algebra.
- If $A = A^{(1)}_{\ell}$ or $A = C^{(1)}_{\ell}$, we may deform the Fock space to the deformed Fock space which is a $U_q(\mathfrak{g}(A))$-module.
- Moreover, we may define the higher level Fock space associated with a multi-charge $(s_1, \ldots, s_r)$:

$$\mathcal{F}_{(s_1,\ldots,s_r)} = \mathcal{F}_{s_1} \otimes \cdots \otimes \mathcal{F}_{s_r}.$$

- The vacuum vector generates the highest weight module $V_q(\Lambda)$ where $\Lambda = \Lambda_{s_1} + \cdots + \Lambda_{s_r}$ is the corresponding dominant integral weight.

Integrable highest weight modules and Hecke algebras

The level one and level two Fock spaces in type $A_\ell^{(1)}$ are those Fock spaces we have been using for studying Hecke algebras associated with irreducible Weyl groups of classical type, which are important algebras in Lie theory.

The use of the Fock spaces will be explained later. The algebras were generalized to cyclotomic Hecke and cyclotomic quiver Hecke algebras.

In the next several slides, we define the cyclotomic quiver Hecke algebra, which categorify the integrable highest weight module $V_q(\Lambda)$ over the quantized enveloping algebra $U_q(\mathfrak{g}(A))$, where $A = (a_{ij})_{i,j \in I}$ is a symmetrizable Cartan matrix.

Definition of cyclotomic quiver Hecke algebras

- Let P be the weight lattice.
- Let $\Pi = \{\alpha_i\}_{i \in I}$ be the set of simple roots.
- We fix a dominant integral weight $\Lambda \in P$.
- Let K be a field.
- We fix a system of polynomials $Q_{i,j}(u,v) \in K[u,v]$, for $i,j \in I$:

$$Q_{i,j}(u,v) = \begin{cases} \sum_{p(\alpha_i,\alpha_i)+q(\alpha_j,\alpha_j)+2(\alpha_i,\alpha_j)=0} t_{i,j;p,q} u^p v^q & \text{if } i \neq j, \\ 0 & \text{if } i = j, \end{cases}$$

where $t_{i,j;p,q} \in K$ are such that $t_{i,j;-a_{ij},0} \in K^\times$ and

$$Q_{i,j}(u,v) = Q_{j,i}(v,u).$$

Definition (cont'd)

The cyclotomic quiver Hecke algebra $R^\Lambda(n)$ associated with polynomials $(Q_{i,j}(u,v))_{i,j\in I}$ and a dominant integral weight $\Lambda \in P$ is the $\mathbb{Z}$-graded unital associative K-algebra generated by

$$\{e(\nu) \mid \nu \in I^n\} \cup \{x_1, \dots, x_n\} \cup \{\psi_1, \dots, \psi_{n-1}\}$$

subject to the following relations.

$$e(\nu)e(\nu') = \delta_{\nu,\nu'} e(\nu)$$

$$\sum_{\nu \in I^n} e(\nu) = 1$$

$$x_r e(\nu) = e(\nu) x_r, \quad x_r x_s = x_s x_r$$

$$\psi_r e(\nu) = e(s_r \nu) \psi_r$$

Definition (cont'd)

$$
\begin{aligned}
\psi_r x_s &= x_s \psi_r \quad (\text{if } s \neq r, r+1) \\
\psi_r \psi_s &= \psi_s \psi_r \quad (\text{if } r \neq s \pm 1) \\
x_r \psi_r e(\nu) &= (\psi_r x_{r+1} - \delta_{\nu_r, \nu_{r+1}}) e(\nu) \\
x_{r+1} \psi_r e(\nu) &= (\psi_r x_r + \delta_{\nu_r, \nu_{r+1}}) e(\nu) \\
\psi_r^2 e(\nu) &= Q_{\nu_r, \nu_{r+1}}(x_r, x_{r+1}) e(\nu)
\end{aligned}
$$

$$
\begin{aligned}
&(\psi_{r+1}\psi_r\psi_{r+1} - \psi_r\psi_{r+1}\psi_r)e(\nu) \\
&\qquad = \begin{cases} \frac{Q_{\nu_r,\nu_{r+1}}(x_r,x_{r+1}) - Q_{\nu_r,\nu_{r+1}}(x_{r+2},x_{r+1})}{x_r - x_{r+2}} e(\nu) & \text{if } \nu_r = \nu_{r+2}, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

and the cyclotomic condition $x_1^{\langle \alpha_{\nu_1}^\vee, \Lambda \rangle} e(\nu) = 0$.

It is known that $R^\Lambda(n)$ are symmetric algebras.

The polynomials $Q_{i,j}(u, v)$

The quiver Hecke algebras for $A = A^{(2)}_{2\ell}$, $D^{(2)}_{\ell+1}$ and $C^{(1)}_{\ell}$ do not depend on the choice of the polynomials $Q_{i,j}(u, v)$.

For $A^{(1)}_{\ell}$, we may assume that the polynomials are

$$\mathcal{Q}_{0,1}(u, v) = u^2 + \lambda uv + v^2,$$

if $\ell = 1$, and

$$\begin{aligned} \mathcal{Q}_{i,i+1}(u, v) &= u + v \ \ (0 \le i \le \ell - 1), \\ \mathcal{Q}_{\ell,0}(u, v) &= u + \lambda v, \\ \mathcal{Q}_{i,j}(u, v) &= 1 \ \ (j \not\equiv i \pm 1 \mod (\ell + 1)), \end{aligned}$$

if $\ell \ge 2$. Here λ is a parameter and the isomorphism class of $R^{\Lambda}(n)$ depends on λ in general.

The affine quiver Hecke algebras

The algebra $R^{\Lambda}(n)$ is finite dimensional. If we drop the last cyclotomic condition, we denote the algebra by $R(n)$.

I wish to call it the affine quiver Hecke algebra. It is infinite dimensional.

Remark 3.1

Recently, they are used in the monoidal categorification of cluster algebras $A_q(\mathfrak{n}(w))$ by Kang, Kashiwara, Kim and Oh.

The grading and block decompositions

- The algebra $R^\Lambda(n)$ and $R(n)$ are given $\mathbb{Z}$-grading as follows.

$$\deg(e(\nu)) = 0, \ \ \deg(x_r e(\nu)) = (\alpha_{\nu_r}, \alpha_{\nu_r}),$$
$$\deg(\psi_s e(\nu)) = -(\alpha_{\nu_s}, \alpha_{\nu_{s+1}}).$$

- For $\beta \in Q^+$ with $\mathrm{ht}(\beta) = n$, set

$$I^\beta = \{\nu \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta\}.$$

Then $e(\beta) = \sum_{\nu \in I^\beta} e(\nu)$ is a central idempotent and

$$R^\Lambda(n) = \bigoplus_{\substack{\beta \in Q^+ \\ \mathrm{ht}(\beta)=n}} R^\Lambda(\beta), \text{ where } R^\Lambda(\beta) = R^\Lambda(n)e(\beta).$$

Categorification of integrable modules

Theorem 3.2 (Kang-Kashiwara)

Let A be a symmetrizable Cartan matrix.

- *The complexified Grothendieck group of finite dimensional $R^\Lambda(\beta)$-modules may be identified with the weight space $V_q(\Lambda)_{\Lambda-\beta}$ of the highest weight $U_q(\mathfrak{g}(A))$-module $V_q(\Lambda)$.*
- *There exist exact functors E_i and F_i from the module category of $R^\Lambda(\beta)$ to the module category of $R^\Lambda(\beta \pm \alpha_i)$ which descend to the action of Chevalley generators e_i and f_i on $V_q(\Lambda)$.*

Block algebras and weight spaces of integrable modules

Theorem 3.3 (Lyle-Mathas)

If $A = A_\ell^{(1)}$ and $\lambda = (-1)^{\ell+1}$ if $\ell \geq 2$, $\lambda = -2$ if $\ell = 1$ then $R^\Lambda(\beta)$ is an indecomposable algebra. Namely, it is a block algebra of $R^\Lambda(n)$.

Thus, the weight spaces of $V_q(\Lambda)$ correspond to the block algebras of $R^\Lambda(n)$, where $n = 0, 1, \ldots,$ in this case.

Conjecture 3.4

The algebras $R^\Lambda(\beta)$ are indecomposable for any affine Lie type.

Cyclotomic Hecke algebras (AK and Broué-Malle)

Recall that the cyclotomic Hecke algebra $H_{G(m,1,n)}(q, Q_1, \ldots, Q_m)$ is defined by generators $T_0, T_1, \ldots, T_{n-1}$ subject to the relations

$$(T_0 - Q_1)\cdots(T_0 - Q_m) = 0, \ \ (T_i - q)(T_i + 1) = 0 \ (1 \le i \le n-1)$$
$$(T_0 T_1)^2 = (T_1 T_0)^2, \ \ T_i T_j = T_j T_i \ (j \ne i \pm 1)$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \ (1 \le i \le n-2)$$

Theorem 3.5 (Brundan-Kleshchev)

Suppose that q is a primitive $(\ell+1)^{th}$ root of unity and $Q_i = q^{s_i}$. We set $A = A_\ell^{(1)}$, $\Lambda = \Lambda_{s_1} + \cdots + \Lambda_{s_m}$ and $\lambda = (-1)^{\ell+1}$. Then

$$R^\Lambda(n) \simeq H_{G(m,1,n)}(q, Q_1, \ldots, Q_m).$$

The similar result holds for the degenerate cyclotomic Hecke algebra.

Dimension formulas: $A^{(2)}_{2\ell}$

Using the embedding of $V(\Lambda_0)$ into the Fock space, we may deduce the dimension formulas for $A^{(2)}_{2\ell}$ and $D^{(2)}_{\ell+1}$.

Theorem 3.6 (A-Park)

We assume that the Cartan matrix is $A^{(2)}_{2\ell}$. Then,

$$\dim e(\nu')R^{\Lambda_0}(n)e(\nu) = \sum_{\lambda\vdash n} 2^{-\langle d,\mathrm{wt}(\lambda)\rangle - l(\lambda)} K(\lambda,\nu')K(\lambda,\nu),$$

$$\dim R^{\Lambda_0}(\beta) = \sum_{\lambda\vdash n,\ \mathrm{wt}(\lambda)=\Lambda_0-\beta} 2^{-\langle d,\mathrm{wt}(\lambda)\rangle - l(\lambda)} |\mathrm{ST}(\lambda)|^2,$$

$$\dim R^{\Lambda_0}(n) = \sum_{\lambda\vdash n} 2^{-\langle d,\mathrm{wt}(\lambda)\rangle - l(\lambda)} |\mathrm{ST}(\lambda)|^2,$$

where $\nu,\nu'\in I^n$ and $K(\lambda,\nu)$ is the number of standard tableaux whose residue sequence is ν.

Dimension formulas: $D^{(2)}_{\ell+1}$

Theorem 3.7 (A-Park)

We assume that the Cartan matrix is $D^{(2)}_{\ell+1}$. Then,

$$\dim e(\nu')R^{\Lambda_0}(n)e(\nu) = \sum_{\lambda\vdash n} 2^{-\langle \mathsf{d},\mathrm{wt}(\lambda)\rangle - l(\lambda)} K(\lambda,\nu')K(\lambda,\nu),$$

$$\dim R^{\Lambda_0}(\beta) = \sum_{\lambda\vdash n,\ \mathrm{wt}(\lambda)=\Lambda_0-\beta} 2^{-\langle \mathsf{d},\mathrm{wt}(\lambda)\rangle - l(\lambda)} |\mathrm{ST}(\lambda)|^2,$$

$$\dim R^{\Lambda_0}(n) = \sum_{\lambda\vdash n} 2^{-\langle \mathsf{d},\mathrm{wt}(\lambda)\rangle - l(\lambda)} |\mathrm{ST}(\lambda)|^2.$$

Oh and Park have obtained graded dimension formulas for $A^{(2)}_{2\ell}$ and $D^{(2)}_{\ell+1}$ by using Young walls, generalization of Young diagrams.

Graded dimension formulas: $A_\ell^{(1)}$ and $C_\ell^{(1)}$

For $A_\ell^{(1)}$ and $C_\ell^{(1)}$, we use the embedding of $V_q(\Lambda)$ to the deformed Fock space to deduce graded dimension formulas. We may define the statistics $\deg(T)$ and $K_q(\lambda,\nu)$ is the sum of $q^{\deg(T)}$ over $\mathrm{ST}(\lambda,\nu)$.

Theorem 3.8 (Brundan-Kleshchev-Wang, A-Park)

Assume that the Cartan matrix is $A_\ell^{(1)}$ or $C_\ell^{(1)}$. For $\nu,\nu'\in I^n$, we have

$$\dim_q e(\nu)R^\Lambda(\beta)e(\nu') = \sum_{\lambda\vdash n, \mathrm{wt}(\lambda)=\Lambda-\beta} K_q(\lambda,\nu)K_q(\lambda,\nu'),$$

$$\dim_q R^\Lambda(\beta) = \sum_{\lambda\vdash n, \mathrm{wt}(\lambda)=\Lambda-\beta} K_q(\lambda)^2,$$

$$\dim_q R^\Lambda(n) = \sum_{\lambda\vdash n} K_q(\lambda)^2,$$

where $\dim_q M := \sum_{k\in\mathbb{Z}} \dim(M_k)q^k$ *for* $M=\oplus_{k\in\mathbb{Z}} M_k$.

Chuang-Rouquer derived equivalence

Since $V(\Lambda)$ is an integrable module, there exists action of the affine Weyl group. For example, if the Cartan matrix is $A_\ell^{(1)}$ then the affine Weyl group is the affine symmetric group. Let $e = \ell + 1$. Then

$$\left\langle \{s_i\}_{i \in \mathbb{Z}/e\mathbb{Z}} \mid s_i s_j s_i = s_j s_i s_j \ (j - i \pm 1 \in e\mathbb{Z}),\ s_i s_j = s_j s_i \ (\text{otherwise}) \right\rangle.$$

Theorem 3.9 (Chuang-Rouquier)

Suppose that two weights $\Lambda - \beta$ and $\Lambda - \beta'$ belong to the same affine Weyl group orbit. Then, $R^\Lambda(\beta)$ and $R^\Lambda(\beta')$ are derived equivalent.

In the application to cyclotomic Hecke algebras, e is identified with the quantum characteristic $\min\{k \in \mathbb{N} \mid 1 + q + \cdots + q^{k-1} = 0 \text{ in } K\}$.

Tame block algebras of Hecke algebras

We use the graded dimension formula in type $A^{(1)}_{e-1}$ to analyze the idempotent truncation of block algebras $R^\Lambda(\beta)$. The advantage of the KLR generators compared with the classical Coxeter generators is that it is easy to construct idempotents and compute the Gabriel quiver of the idempotent truncation.

Suppose that the characteristic of the base field is odd. Computation based on the above tools plus various results such as Ohmatsu's theorem, Rickard's star theorem, criterion of tilting discreteness by Adachi, Aihara and Chan for Brauer graph algebras, we may show that the basic algebras of tame block algebras are very restricted. Here, the cellularity in the sense of Graham and Lehrer plays an important role.

Note that block algebras are cellular by old results by Dipper, James and Murphy for type A and B, by Geck for type D.

Tame block algebras of Hecke algebras (cont'd)

Theorem 4.1 (A)

We consider block algebras of Hecke algebras of classical type over an algebraically closed field of odd characteristic and $q \neq 1$, $e \geq 2$.

If it is finite, then it is Morita equivalent to a Brauer line algebra. If it is infinite-tame, then it is Morita equivalent to one of the algebras below.

(1) *In type A or D, we must have $e = 2$ and it is a Brauer graph algebra whose Brauer graph is $\textcircled{2} - \bigcirc - \textcircled{2}$ or $\textcircled{2} - \textcircled{2} - \bigcirc$.*

(2) *In type B with $e = 2$, it is either one of the Brauer graph algebras in (1), or the symmetric Kronecker algebra, which is the Brauer graph algebra with one non-exceptional vertex and one loop. Otherwise, we must have $e \geq 4$ is even and $Q = -1$ and it is the Brauer graph algebra whose Brauer graph is $\textcircled{2} - \textcircled{2} - \textcircled{2}$.*

Specht modules in affine type C

The following is another application of the categorification.

Theorem 4.2 (A-Park-Speyer)

We may define Specht modules S^λ, for multi-partitions $\lambda \vdash n$. If n is small enough such that the height n part of $V(\Lambda)$ for $C_\ell^{(1)}$ and C_∞ are the same, then $\mathrm{ST}(\lambda)$ form a basis of S^λ and we have the graded character formula:

$$\mathrm{ch}_q S^\lambda = \sum_{T \in \mathrm{ST}(\lambda)} q^{\deg(T)} \mathrm{res}(T).$$

Conjecture 4.3

Supose that the Cartan matrix is of affine type C. Then, the Specht modules give a cellular algebra structure on $R^\Lambda(n)$.

Thank you for your attention.