Wide subcategories and lattices of torsion classes

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Torsion pairs

Let $\mathcal{A}$ be an (ess. small) abelian length category.

- Any object $X \in \mathcal{A}$ has a composition series $0 = X_0 \subset X_1 \subset \cdots \subset X_n = X$ with X_i/X_{i-1}: simple.

Definition [Dickson]

Let $\mathcal{T}, \mathcal{F} \subset \mathcal{A}$. $(\mathcal{T}, \mathcal{F})$ is called a torsion pair in $\mathcal{A}$ if

- $\mathcal{F} = \mathcal{T}^{\perp} := \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, X) = 0\}$,
- $\mathcal{T} = {}^{\perp}\mathcal{F} := \{X \in \mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(X, \mathcal{F}) = 0\}$.

Or equivalently,

- $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$,
- $\forall X \in \mathcal{A}, \exists (0 \to X' \to X \to X'' \to 0)$: exact, $X' \in \mathcal{T}, X'' \in \mathcal{F}$.

Lattice of torsion classes

Definition

$\mathcal{T} \subset \mathcal{A}$: torsion class $:\Longleftrightarrow$ $(\mathcal{T}, \mathcal{T}^{\perp})$: torsion pair

$\Longleftrightarrow$ $\mathcal{T}$ is closed under factor obj's, extensions

- $\operatorname{tors}\mathcal{A} := \{$all torsion classes in $\mathcal{A}\}$: poset by $\subset$.
- For any $\mathcal{X} \subset \mathcal{A}$, there exists $\mathsf{T}(\mathcal{X}) :=$ (the smallest torsion class containing $\mathcal{X}$).

Proposition

$\operatorname{tors}\mathcal{A}$ is a complete lattice with meets and joins

$$\bigwedge_{\mathcal{T}\in S} \mathcal{T} = \bigcap_{\mathcal{T}\in S} \mathcal{T}, \quad \bigvee_{\mathcal{T}\in S} \mathcal{T} = \mathsf{T}\left(\bigcup_{\mathcal{T}\in S} \mathcal{T}\right) \quad (S \subset \operatorname{tors}\mathcal{A}).$$

Wide intervals

For intervals $[\mathcal{U}, \mathcal{T}]$ in tors $\mathcal{A}$ (with $\mathcal{U} \subset \mathcal{T}$), $\mathcal{U}^{\perp} \cap \mathcal{T}$ gives the "difference" of $\mathcal{U}$ and $\mathcal{T}$.
Today, we deal with the following nice intervals.

Definition [AP]

An interval $[\mathcal{U}, \mathcal{T}]$ in tors $\mathcal{A}$ is called a wide interval if $\mathcal{U}^{\perp} \cap \mathcal{T}$ is a wide subcategory of $\mathcal{A}$.

$\mathcal{W} \subset \mathcal{A}$ is called a wide subcategory if
$\mathcal{W}$ is closed under kernels, cokernels, extensions.

- A wide subcat. $\mathcal{W}$ is an abelian length category.
- There exists a bij. $\{$wide subcat.$\} \leftrightarrow \{$semibricks$\}$.
  - a semibrick = a set of pairwise Hom-orthogonal bricks.
  - $\mathcal{W} \mapsto \{$the simple objects of $\mathcal{W}\}$, Filt $\mathcal{S} \leftarrow\!\shortmid \mathcal{S}$.

Brick labeling

Two torsion classes $\mathcal{U} \subset \mathcal{T}$ are said to be adjacent if $\mathcal{U} \neq \mathcal{T}$ and $\nexists \mathcal{V} \in \operatorname{tors} \mathcal{A}, \mathcal{U} \subsetneq \mathcal{V} \subsetneq \mathcal{T}$.

Definition

The Hasse quiver of $\operatorname{tors} \mathcal{A}$ is defined as follows.

- The vertices are the elements of $\operatorname{tors} \mathcal{A}$.
- Write an arrow $\mathcal{T} \to \mathcal{U}$ if $\mathcal{U} \subsetneq \mathcal{T}$ are adjacent.

Proposition [Demonet–Iyama–Reading–Reiten–Thomas]

For any arrow $q \colon \mathcal{T} \to \mathcal{U}$, $[\mathcal{U}, \mathcal{T}]$ is a wide interval, and $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$ has a unique simple object S_q, so we label $q \colon \mathcal{T} \to \mathcal{U}$ by the brick S_q.

τ-tilting reduction

Let A be a fin. dim. alg. over a field K, and $\mathcal{A} = \operatorname{mod} A$. For $N \in \operatorname{mod} A$ and $Q \in \operatorname{proj} A$, (N, Q) is a τ-rigid pair if $\operatorname{Hom}_A(N, \tau N) = 0$ and $\operatorname{Hom}_A(Q, N) = 0$.

Theorem [Jasso, DIRRT]

For a τ-rigid pair (N, Q), set

$$\mathcal{U} := \operatorname{Fac} N, \quad \mathcal{T} := N^{\perp} \cap {}^{\perp}(\tau N) \cap Q^{\perp}.$$

(1) $[\mathcal{U}, \mathcal{T}]$ is a wide interval ($\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$ is wide).

(2) $[\mathcal{U}, \mathcal{T}] \cong \operatorname{tors} \mathcal{W}$ as complete lattices, where

$$\mathcal{V} \mapsto \mathcal{U}^{\perp} \cap \mathcal{V}, \quad \mathsf{T}(\mathcal{U} \cup \mathcal{X}) \leftarrow\!\shortmid \mathcal{X}.$$

(3) The bijections in (2) preserve brick labeling.

$\mathcal{W} \cong \operatorname{mod} C$ for some fin. dim. alg C.

Main result

Theorem 1 [AP]

Let $[\mathcal{U},\mathcal{T}]$ be a wide interval in tors $\mathcal{A}$, $\mathcal{W} := \mathcal{U}^{\perp} \cap \mathcal{T}$.

(1) $[\mathcal{U},\mathcal{T}] \cong$ tors $\mathcal{W}$ as complete lattices, where

$$\mathcal{V} \mapsto \mathcal{U}^{\perp} \cap \mathcal{V} =: \Phi(V),$$

$$\Phi^{-1}(\mathcal{X}) = \mathsf{T}(\mathcal{U} \cup \mathcal{X}) \leftarrow\!\shortmid \mathcal{X}.$$

(2) The bijection Φ preserves brick labeling: the label of $\mathcal{V}_1 \to \mathcal{V}_2$ is the label of $\Phi(\mathcal{V}_1) \to \Phi(\mathcal{V}_2)$.

(3) The following sets coincide:

- **(a)** The set of the labels of the arrows from $\mathcal{T}$ in $[\mathcal{U},\mathcal{T}]$.
- **(b)** The set of the labels of the arrows to $\mathcal{U}$ in $[\mathcal{U},\mathcal{T}]$.
- **(c)** The set of the simple objects of $\mathcal{W}$.

"Not τ-tilting" example

Let $K = \overline{K}$, $A = K(1 \rightrightarrows 2)$ and $\mathcal{A} = \operatorname{mod} A$.
We set $\mathcal{U}, \mathcal{T} \in \operatorname{tors} \mathcal{A}$ by

- $\mathcal{U}$:= add{all preinjective modules},
- $\mathcal{T}$:= add{all regular, preinjective modules}.

Then, $[\mathcal{U}, \mathcal{T}]$ is a wide interval with

$$\begin{aligned}
\mathcal{W} &= \operatorname{add}\{\text{all regular modules}\} \\
&= \operatorname{Filt}\{M_\lambda \mid \lambda \in \mathbb{P}^1(K)\} \\
&\left(M_\lambda := \begin{tikzcd} K \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] & K \end{tikzcd} \ (\lambda = (a:b) \in \mathbb{P}^1(K)) \right) \\
&= \bigoplus_{\lambda \in \mathbb{P}^1(K)} \operatorname{Filt} M_\lambda.
\end{aligned}$$

"Not τ-tilting" example

$[\mathcal{U}, \mathcal{T}]$ is a wide interval with

$$\mathcal{W} = \bigoplus_{\lambda \in \mathbb{P}^1(K)} \operatorname{Filt} M_\lambda .$$

Since $\operatorname{tors}(\operatorname{Filt} M_\lambda) = \{\operatorname{Filt} M_\lambda, \{0\}\}$,

$$[\mathcal{U}, \mathcal{T}] \cong \operatorname{tors} \mathcal{W} \cong \prod_{\lambda \in \mathbb{P}^1(K)} \operatorname{tors}(\operatorname{Filt} M_\lambda) \cong 2^{\mathbb{P}^1(K)}.$$

For $X \in 2^{\mathbb{P}^1(K)}$, the associated torsion class in $[\mathcal{U}, \mathcal{T}]$ is

$$\mathcal{V}_X := \mathsf{T}(\mathcal{U} \cup \{M_\lambda \mid \lambda \in X\}) \in [\mathcal{U}, \mathcal{T}].$$

Any arrow in $[\mathcal{U}, \mathcal{T}]$ is of the form

$$\mathcal{V}_{X \cup \{\lambda\}} \xrightarrow{\text{label: } M_\lambda} \mathcal{V}_X \quad (X \in 2^{\mathbb{P}^1(K)},\ \lambda \in \mathbb{P}^1(K) \setminus X).$$

Characterization of wide intervals (1)

For any interval $[\mathcal{U},\mathcal{T}]$ in tors $\mathcal{A}$, we set

$$[\mathcal{U},\mathcal{T}]^{+} := \{\mathcal{T}\} \cup \{\mathcal{V} \in [\mathcal{U},\mathcal{T}] \mid \exists(\mathcal{T} \to \mathcal{V})\text{: arrow}\},$$
$$[\mathcal{U},\mathcal{T}]^{-} := \{\mathcal{U}\} \cup \{\mathcal{V} \in [\mathcal{U},\mathcal{T}] \mid \exists(\mathcal{V} \to \mathcal{U})\text{: arrow}\}.$$

Theorem 2 [AP]

For any interval $[\mathcal{U},\mathcal{T}]$ in tors $\mathcal{A}$, TFAE.

(a) $[\mathcal{U},\mathcal{T}]$ is a wide interval.

(b) $[\mathcal{U},\mathcal{T}]$ is a join interval, i.e. $\mathcal{T} = \bigvee_{\mathcal{V}\in[\mathcal{U},\mathcal{T}]^{-}} \mathcal{V}$.

(c) $[\mathcal{U},\mathcal{T}]$ is a meet interval, i.e. $\mathcal{U} = \bigwedge_{\mathcal{V}\in[\mathcal{U},\mathcal{T}]^{+}} \mathcal{V}$.

Characterization of wide intervals (2)

Question

How many wide intervals $[\mathcal{U}, \mathcal{T}]$ exist for $\mathcal{T} \in \operatorname{tors} \mathcal{A}$?

Theorem 3 [AP]

Fix $\mathcal{T} \in \operatorname{tors} \mathcal{A}$ and $\mathcal{L} := \{\text{all labels of arrows from } \mathcal{T}\}$.

(1) $\mathcal{L}$ is a semibrick with $\operatorname{Filt} \mathcal{L} = \alpha(\mathcal{T})$, where

$$\alpha(\mathcal{T}) := \{X \in \mathcal{T} \mid \forall Y \in \mathcal{T}, \forall f \colon Y \to X, \operatorname{Ker} f \in \mathcal{T}\}.$$

(2) There exists a bijection

$$2^{\mathcal{L}} \to \{\mathcal{U} \in \operatorname{tors} \mathcal{A} \mid [\mathcal{U}, \mathcal{T}] \colon \text{wide interval}\}$$
$$\mathcal{S} \mapsto \mathcal{T} \cap {}^{\perp}\mathcal{S} =: \mathcal{U}_{\mathcal{S}}$$

and $(\mathcal{U}_{\mathcal{S}})^{\perp} \cap \mathcal{T} = \operatorname{Filt} \mathcal{S} \subset \alpha(\mathcal{T})$: Serre.

Widely generated torsion classes

Theorem [Marks–Šťovíček]

If $\mathcal{W}$ is a wide subcategory of $\mathcal{A}$, then $\alpha(\mathsf{T}(\mathcal{W})) = \mathcal{W}$.

Corollary [AP] (cf. [Barnard–Carroll–Zhu])

For $\mathcal{T} \in \operatorname{tors} \mathcal{A}$, TFAE.

(a) $\exists \mathcal{W} \subset \mathcal{A}$: a wide subcat., $\mathcal{T} = \mathsf{T}(\mathcal{W})$ (widely generated torsion classes).

(b) $\mathcal{T} = \mathsf{T}(\alpha(\mathcal{T}))$.

(c) $\mathcal{T} = \mathsf{T}(\{\text{all labels of arrows from } \mathcal{T}\})$.

(d) $\forall \mathcal{U} \subsetneq \mathcal{T}, \exists (\mathcal{T} \to \mathcal{U}')$: arrow, $\mathcal{U} \subset \mathcal{U}'$.

Thank you for your attention.