Functors between higher cluster categories of type $\mathbb{A}$

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Higher cluster categories

Let k be a field, H a finite-dimensional hereditary k-algebra, and $m \geqslant 1$.

The *m-cluster category* of H is the orbit category

$$\mathscr{C}^m(H) = \frac{\mathscr{D}^{\mathrm{b}}(\operatorname{mod} H)}{\nu \circ [-m-1]}\,.$$

- Triangulated;
- $m = 1$ recovers classical BMRRT cluster categories;
- cluster-tilting objects categorify certain combinatorics;
- $|\operatorname{ind} \mathscr{C}^m(H)| < \infty \Leftrightarrow H$ is of Dynkin type.

Geometric description in type $\mathbb{A}$

[Baur–Marsh 2008], [Caldero–Chapoton–Schiffler 2006] for $m = 1$

Let

$l, m \geqslant 1, \quad N = m(l+1)+2,$

P_N : a polygon with N vertices.

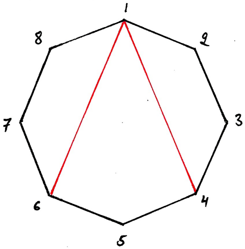
- m-diagonal $[x, y]$ in P_N:
  a diagonal between vertices x and y, such that $|x-y| > 1$ and $m|(|x-y|-1)$ (modulo N).
- Every non-crossing collection of m-diagonals can be refined into an "$(m+2)$-angulation" of P_N.

P_N	$\mathscr{C}^m(\mathbb{A}_l)$
m-diagonal	indecomposable object
$(m+2)$-angulation	m-cluster-tilting object
ρ^{-m}	τ
$[x, y] \to [x, y+m]$	irreducible map

ρ = rotation one step in P_N.

Example

$m = l = 2, \quad N = 8$



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$$\begin{tikzcd}[column sep=small]
47 \arrow[dr] & & 16 \arrow[dr] & & 38 \arrow[dr] & & 25 \arrow[dr] & & 47 \\
& 14 \arrow[ur] & & 36 \arrow[ur] & & 58 \arrow[ur] & & 26 \arrow[ur] &
\end{tikzcd}$$

Ambiguity

If $l, m,\ l', m' \geqslant 1$, and

$$\frac{m'}{m} = \frac{l+1}{l'+1} \in \mathbb{Z}$$

then

$$(l'+1)m' + 2 = (l+1)m + 2 = N$$

and

$$m' | (|x-y|-1) \Rightarrow m | (|x-y|-1)$$

i.e., every m'-diagonal is also an m-diagonal (but not vice versa).

$$\rightsquigarrow \quad \mathrm{ob}(\mathscr{C}^{m'}(\mathbb{A}_{l'})) \hookrightarrow \mathrm{ob}(\mathscr{C}^{m}(\mathbb{A}_{l}))$$

Does it come from an embedding of categories?

No, it doesn't

$m' = l' = 2, \quad m = 1, l = 5, \qquad N = 8$

$$\mathscr{C}^2(\mathbb{A}_2)\ :\quad \begin{tikzcd}[column sep=small]
& 16 \arrow[dr] & & 38 \arrow[dr] & & 25 \arrow[dr] & & 47 \arrow[dr] & \\
14 \arrow[ur] & & 36 \arrow[ur] & & 58 \arrow[ur] & & 26 \arrow[ur] & & 14
\end{tikzcd}$$

$$\mathscr{C}^1(\mathbb{A}_5)\ :\quad \begin{tikzcd}[column sep=small]
& & & & 17 \arrow[dr] & & 28 \arrow[dr] & & 13 \arrow[dr] & & & & \\
& & & 16 \arrow[ur] \arrow[dr] & & 27 \arrow[ur] \arrow[dr] & & 38 \arrow[ur] \arrow[dr] & & 14 \arrow[dr] & & & \\
& & 15 \arrow[ur] \arrow[dr] & & 26 \arrow[ur] \arrow[dr] & & 37 \arrow[ur] \arrow[dr] & & 48 \arrow[ur] \arrow[dr] & & 15 \arrow[dr] & & \\
& 14 \arrow[ur] \arrow[dr] & & 25 \arrow[ur] \arrow[dr] & & 36 \arrow[ur] \arrow[dr] & & 47 \arrow[ur] \arrow[dr] & & 58 \arrow[ur] \arrow[dr] & & 16 \arrow[dr] & \\
13 \arrow[ur] & & 24 \arrow[ur] & & 35 \arrow[ur] & & 46 \arrow[ur] & & 57 \arrow[ur] & & 68 \arrow[ur] & & 17
\end{tikzcd}$$

We need some more space.

Theorem [van Roozendaal 2019]

Assume that $\dfrac{m'}{m} = \dfrac{l+1}{l'+1} =: a \in \mathbb{Z}$.

Then there exists a full and faithful functor

$$F : \mathscr{C}^{m'}(\mathbb{A}_{l'}) \to \mathscr{D}^{\mathrm{b}}(\operatorname{mod} k\mathbb{A}_l)/(\nu \circ [-m-1])^{a}$$

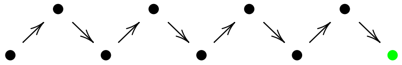
such that

$$\begin{tikzcd}
\mathscr{C}^{m'}(\mathbb{A}_{l'}) \arrow[r, "F"] \arrow[dr, dotted, "\text{object embedding}"'] & \dfrac{\mathscr{D}^{\mathrm{b}}(\operatorname{mod} k\mathbb{A}_l)}{(\nu\circ[-m-1])^{a}} \arrow[d, "\text{covering}"] \\
 & \mathscr{C}^{m}(\mathbb{A}_l)
\end{tikzcd}$$

Embedding $\mathscr{C}^2(\mathbb{A}_2) \to \mathscr{D}^{\mathrm{b}}(\mathrm{mod}\, k\mathbb{A}_5)/(\nu \circ [-1-1])^2$

$m' = l' = 2,\quad m = 1, l = 5,\quad a = 2$

$\mathscr{C}^2(k\mathbb{A}_2)$:



$\mathscr{D}^{\mathrm{b}}(k\mathbb{A}_5)/(\nu \circ [-2])^2$: