Cohen-Macaulay modules over Yoneda algebras

Λ : ring , M : Λ-module

$\rightsquigarrow$ $\Gamma := \bigoplus_{i \geq 0} \mathrm{Ext}^i_\Lambda(M, M)$ graded ring

Setup ◦ Λ : f.d. alg , representation-finite

◦ $M \in \mathrm{mod}\,\Lambda$: additive generator.

Aim Study such Γ. : call it the Yoneda alg. of Λ.

Rem ◦ Different setup from "Koszul duality" or "Hochschild cohomology".

◦ Γ is f.d. $\Leftrightarrow$ gl. dim $\Lambda < \infty$

◦ In general, Γ is not even Noetherian.

Thm Γ : Yoneda alg of Λ.

(1) Γ is graded coherent.

(2) Γ is 1-Iwanaga-Gorenstein.

(3) There exists a triangle equivalence

$$\underline{\mathrm{CM}}^{\mathbb{Z}}\,\Gamma \simeq \mathrm{D}^b(\mathrm{mod}\,\underline{\Gamma}),$$

wher $\underline{\Gamma}$ is the stable Auslander alg $\underline{\mathrm{End}}_\Lambda(M)$ of Λ.

Def (1) A graded ring R is graded coherent if $\mathrm{mod}^{\mathbb{Z}} R$ and $\mathrm{mod}^{\mathbb{Z}} R^{op}$ are abelian. ($\Leftrightarrow$ $\mathrm{proj}^{\mathbb{Z}} R$ and $\mathrm{proj}^{\mathbb{Z}} R^{op}$ have weak kernels.)

the cat. of finitely presented modules

(2) A gr. coh. ring R is d-Iwanaga-Gorenstein if $\mathrm{id}\,R \leq d$ in $\mathrm{mod}^{\mathbb{Z}} R$ and in $\mathrm{mod}^{\mathbb{Z}} R^{op}$.

(3) R : gr. coh d-IG ring.

$\mathrm{CM}^{\mathbb{Z}} R := \{ X \in \mathrm{mod}^{\mathbb{Z}} R \mid \mathrm{Ext}^{>0}_R(X, R) = 0 \}$

the category of Cohen-Macaulay R-modules;

$\mathrm{CM}^{\mathbb{Z}} R$: Frob , so $\underline{\mathrm{CM}}^{\mathbb{Z}} R$: triangulated.

Big Question

When is there a triangle equivalence $\underline{\mathrm{CM}}^{\mathbb{Z}} R \simeq \mathrm{D}^b(\mathrm{mod}\,A)$ for some algebra A ?

How to study?

■ **Starting point:**

Consider $\mathcal{Y} := \operatorname{add}\{M(i) \mid i \in \mathbb{Z}\} \subset D^b(\operatorname{mod}\Lambda)$

$$\begin{array}{c} \mathcal{Y} \\ \wr\| \\ \operatorname{proj}^{\mathbb{Z}}\Gamma \end{array}$$

since

■ **To prove (1).**

This is a consequence of

Prop $\mathcal{Y} \subset D^b(\Lambda)$: funct. finite. $\left(\rightsquigarrow \mathcal{Y} \simeq \operatorname{proj}^{\mathbb{Z}}\Gamma \text{ has wk. ker's}\right)$

■ **To prove (3)**

Rem $\nexists$ tilt $\in \underline{CM}^{\mathbb{Z}}\Gamma$ (nor silt.)

Use : realization functors.

$\mathcal{T}$: triangulated cat with a t-structure, whose heart is $\mathcal{H}$.

Def A realization functor is a triangle functor $D^b(\mathcal{H}) \to \mathcal{T}$ extending the inclusion $\mathcal{H} \subset \mathcal{T}$.

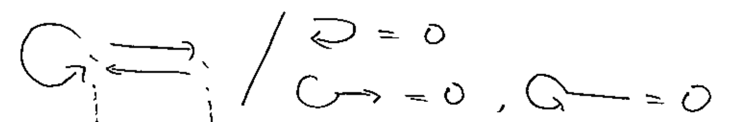
$$\begin{tikzcd} \mathcal{H} \arrow[r, phantom, "\subset"] \arrow[d, phantom, "\cap"] & \mathcal{T} \\ D^b(\mathcal{H}) \arrow[ur, dashed, "\text{real. fct}"'] \end{tikzcd}$$

Fact ([Keller-Vossieck] etc)
Realization functors exist for algebraic triangulated cat's.

Apply this for our $\underline{CM}^{\mathbb{Z}}\Gamma$.

Example

$\Lambda = k[x]/(x^2)$: rep. finite.

$\rightsquigarrow$ $\Gamma =$ 

loop $\in \operatorname{Ext}^1$, simple, proj

By Thm, we have

(1) Γ : graded coherent (This is infact Noeth.)

(2) Γ : 1-IG.

(3) $\underline{CM}^{\mathbb{Z}}\Gamma \simeq D^b(\operatorname{mod} k)$