A remark on graded countable Cohen-Macaulay representation type

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Throughout the talk, k is an algebraically closed field of characteristic 0 and $R = \oplus_{i=0}^{\infty} R_i$ is a (commutative) positively graded affine k-algebra with $R_0 = k$ and $R_+ = \oplus_{i>0}^{\infty} R_i$

- Take a graded Noetherian normalization S of R. That is, $S = k[y_1, \dots, y_n] \subseteq R$ where $n = \dim R$ such that R is a finitely generated graded S-module.

Definition 1

A finitely generated graded R-module M is said to be maximal Cohen-Macaulay (MCM) if M is graded **free** as an S-module. In other words,

$$M \cong S \otimes_k V$$

for some finite dimensional graded k-module V.

Today, we focus on a graded CM algebra of graded countable CM representation type.

Definition 2

We say that $\boldsymbol{R}$ is of graded **“countable”** CM representation type if there are **infinitely but only countably** many isomorphism classes of indecomposable graded MCM modules up to shift.

Definition 3 (Drozd and Tovpyha 2014)

We say that R is of graded **"discrete"** CM representation type if, for any fixed $r > 0$, there are only finitely many isomorphism classes of graded MCM modules with rank r up to shift. Here the rank is taken over S.

Remark.

One can show that if R is of discrete CM representation type then R is of countable CM representation type. Because there is only a countable set of graded MCM R-modules up to isomorphism if R is of discrete. The converse does not hold in general.

Example 4

Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$. Then R is of graded countable CM representation type whose indecomposable MCM R-modules are $I_n = (x, y^n)R$ for $n \geq 0$ up to shift. Note that $\mathrm{rank}_S I_n = 2$ (since $I_n \cong S(-1) \oplus S(-n)$ where $S = k[y]$). But $I_n \ncong I_m$ if $n \neq m$, so that R is not of discrete.

Our motivation of this study is to give the condition that the CM algebras of graded countable CM representation type is of graded discrete representation type.

Main result

Let R be of graded countable CM representation type. Suppose that R is with an isolated singularity. Then R is of graded discrete CM representation type.

Definition 5

We say that R is with an isolated singularity if each graded localization $R_{(\mathfrak{p})}$ is regular for each graded prime ideal $\mathfrak{p}$ with $\mathfrak{p} \neq R_+$.

A variety of graded MCM modules

- Given a graded MCM $\boldsymbol{R}$-module $\boldsymbol{M}$, since $\boldsymbol{M} \cong \boldsymbol{S} \otimes_{\boldsymbol{k}} \boldsymbol{V}$, there exists a "degree $\mathbf{0}$" graded $\boldsymbol{S}$-algebra homomorphism;

$$\exists \alpha \in \mathrm{Hom}_{\boldsymbol{S}}(\boldsymbol{R}, \mathrm{End}_{\boldsymbol{S}}(\boldsymbol{S} \otimes_{\boldsymbol{k}} \boldsymbol{V}))_{\mathbf{0}}.$$

- Remark that

$$\mathrm{Rep}_{\boldsymbol{S}}(\boldsymbol{R}, \boldsymbol{V})(\boldsymbol{k}) := \mathrm{Hom}_{\boldsymbol{S}}(\boldsymbol{R}, \mathrm{End}_{\boldsymbol{S}}(\boldsymbol{S} \otimes_{\boldsymbol{k}} \boldsymbol{V})_{\mathbf{0}}$$

is an algebraic variety over $\boldsymbol{k}$.

Example 6

Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$. Then $S = k[y]$ is a graded Noetherian normalization of R. Set $V = V_0 \oplus V_1$ where $V_0 = V_1 = k$. Then giving a graded S-homorphism $\alpha \in \mathrm{Rep}_S(R, V)(k)$ (*i.e.* giving a graded MCM R-module which is isomorphic to $S \otimes_k V = S \oplus S(-1)$) is equivalent to giving a $\mu_\alpha \in \mathrm{End}_S(S \otimes_k V)_1$ with $\mu_\alpha^2 = 0$. Note that

$$\mathrm{End}_S(S \otimes_k V)_1 = \left\{ \left(\begin{smallmatrix} ay & by^2 \\ c & dy \end{smallmatrix} \right) | a, b, c, d \in k \right\}$$

Hence one can show that

$$\begin{aligned} &\mathrm{Rep}_S(R, V)(k) \\ &= \mathrm{Hom}_{k\text{-alg}}\Big(\frac{k[a, b, c, d]}{(a^2 + bc, ab + bd, ac + bc, bc + d^2)}, k\Big). \end{aligned}$$

- The algebraic group $G_V = \mathrm{Aut}_S(S \otimes_k V)_0$ acts on $\mathrm{Rep}_S(R, V)(k)$ by conjugation.
- There exists 1-1 correspondence;

$$\{G_V\text{-orbits in } \mathrm{Rep}_S(R, V)(k)\}$$
$$\xleftrightarrow{1-1} \{M | M \cong S \otimes_k V \text{as graded } S\text{-modules }\} / \cong$$

- Note that $\mathrm{Rep}_S(R, V)(k)$ parameterizes graded MCM R-modules with fixed **Hilbert series**.

Remark 1 (Dao-Shipman 2014)

The notion $\mathrm{Rep}_S(R, V)(k)$ is introduced by [Dao-Shipman] in 2014. More precisely they define the functor

$$\mathrm{Rep}_S(R, V) : (\text{Commutative } k\text{-algebras}) \to (\text{Sets}).$$

They also show that $\mathrm{Rep}_S(R, V)$ is represented by an affine variety of finite type.

Main theorem

Lemma 7

Let $X \subseteq \mathbb{A}^n(k)$ *be an algebraic set and let* $X_i \subsetneq X$ *be closed subsets with* $\dim X_i < \dim X$. *Then* X *can be* **never** *represented by a* **countable** *union of* X_i.

$$X \neq \bigcup_{i \geq 1} X_i.$$

Theorem 8

Let $X \subseteq \mathbb{A}^n(k)$ *be an algebraic set. Suppose that* $X = \bigcup_{i \geq 1} Y_i$ *where* Y_i *are locally closed and* $Y_i \cap Y_j = \varnothing$ *for* $i \neq j$. *Then*

$$X = \bigcup_{i \geq 1}^{finite} Y_i.$$

Sketch of proof of Theorem 8.

Proof.

- $X = X_1 \cup \cdots \cup X_m$: An irreducible decomposition.
- Then $X_k = \bigcup_{i \geq 1} (X_k \cap Y_i)$. Note that $X_k \cap Y_i$ are locally closed.
- By the lemma, there exists i such that $\dim X_k = \dim X_k \cap Y_i$, so that $X_k = \overline{X_k \cap Y_i}$.
- Since $X_k \cap Y_i$ is open in $\overline{X_k \cap Y_i} = X_k$,

  $$X'_k := X_k \backslash (X_k \cap Y_i)$$

  is closed and $\dim X'_k < \dim X_k$.
- Continuing this procedure, we obtain the result.

□

Remark 2

Let X be a G-variety. Namely X is a variety equipped with an action of the group G. For $x \in X$, we denote by $\mathcal{O}(x)$ the G-orbit of x. Then

- $\mathcal{O}(x)$ is locally closed.
- $\mathcal{O}(x) = \mathcal{O}(y) \Leftrightarrow \overline{\mathcal{O}(x)} = \overline{\mathcal{O}(y)}$ for $x, y \in X$.

Corollary 9

Let R be a graded countable CM representation type. For each finite dimensional graded k-vector space V, there are finitely many isomorphism classes of graded MCM R-modules which are isomorphic to $S \otimes_k V$.

Proof.

$\mathrm{Rep}_S(R, V)(k) = \bigcup_{i \geq 1}^{\mathit{finite}} \overline{\mathcal{O}(x_i)}$ by Theorem 8. □

- Corollary 9 says that, if R is of graded countable representation type, there are only finitely many graded MCM R-modules with a fixed **Hilbert series**.
- It is natural to ask what happens if we fix the **Hilbert polynomial** instead of the **Hilbert series**.

We have the following example.

Example 10

Let $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$ and $I_n = (x, y^n)R$. Then Hilbert polynomials of I_n are 2 for all n and $I_n \ncong I_m$ if $n \neq m$.

The following theorem is due to Dao and Shipman.

Theorem 11 (Dao-Shipman 2014, Theorem 3.1)

Assume that R is with an isolated singularity. For each $r > 0$ there exists $\alpha_r > 0$ such that if M is an indecomposable graded MCM R module with rank r then

$$g_{max}(M) - g_{min}(M) < \alpha_r,$$

where $g_{max}(M) = max\{m|(M/S_+M)m \neq 0\}$ *and* $g_{min}(M) = min\{m|(M/S_+M)m \neq 0\}$.

The theorem says that there are only finitely many finite dimensional graded k-modules which are isomorphic to indecomposable graded MCM R-modules with rank r up to isomorphism and shift. (Note that $M/S_+M \cong V$ as k-modules.)

Corollary 12

Let R be of graded countable CM representation type. Suppose that R is with an isolated singularity. Then R is of graded discrete CM representation type.

Proof.

According to Theorem 11, for each $r > 0$,

$$\{M : \text{MCM } R\text{-modules with rank } r\}/ <\cong, \text{shift}> = \bigcup_{V}^{finite} \mathrm{Rep}_S(R, V)(k).$$

$\square$

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Thank you for your attention.