The Extension Dimension of Abelian Categories

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Introduction

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Given a triangulated category $\mathcal{T}$, Rouquier introduced the dimension $\dim \mathcal{T}$ of $\mathcal{T}$ (2006). Roughly speaking, this dimension is an invariant that measures how quickly the category can be built from one object.

Let Λ be an artin algebra. Let $\mathrm{mod}\,\Lambda$ be the category of finitely generated right Λ-modules and let $D^b(\mathrm{mod}\,\Lambda)$ be the bounded derived category of $\mathrm{mod}\,\Lambda$ respectively. The upper bounds for the dimension of $D^b(\mathrm{mod}\,\Lambda)$ can be given in terms of the Loewy length $\mathrm{LL}(\Lambda)$ and the global dimension $\mathrm{gl.dim}\,\Lambda$ of Λ.

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Let Λ be an artin algebra. Then

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As an analogue of the dimension of triangulated categories, the (extension) dimension $\dim \mathcal{A}$ of an abelian category $\mathcal{A}$ was introduced by Beligiannis (2008).

Let Λ be an artin algebra. Note that the representation dimension $\operatorname{rep.dim} \Lambda$ of Λ is at most two (that is, Λ is of finite representation type) if and only if $\dim \operatorname{mod} \Lambda = 0$ (Beligiannis, 2008). So, like $\operatorname{rep.dim} \Lambda$, the extension dimension $\dim \operatorname{mod} \Lambda$ is also an invariant that measures how far Λ is from having finite representation type.

It was proved that $\dim \operatorname{mod} \Lambda \leq \operatorname{LL}(\Lambda) - 1$ (Beligiannis, 2008), which is a semi-counterpart of the above result of Rouquier.

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For a subclass $\mathcal{U}$ of $\mathcal{A}$, we use $\mathrm{add}\,\mathcal{U}$ to denote the subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of objects in $\mathcal{U}$.

Let $\mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_n$ be subcategories of $\mathcal{A}$. Define $\mathcal{U}_1 \diamond \mathcal{U}_2 := \mathrm{add}\{A \in \mathcal{A} \mid$ there exists an exact sequence

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For any subcategory $\mathcal{X}$ of $\mathcal{A}$, define

$$\textbf{size}_{\mathcal{A}}\,\mathcal{X} := \inf\{n \geq 0 \mid \mathcal{X} \subseteq \langle T \rangle_{n+1} \text{ with } T \in \mathcal{A}\},$$

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2.2. Radical layer length and torsion pairs

Let $\mathcal{C}$ be a **length-category**, that is, $\mathcal{C}$ is an abelian, skeletally small category and every object of $\mathcal{C}$ has a finite composition series. We use $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ to denote the category of all additive functors from $\mathcal{C}$ to $\mathcal{C}$, and use rad to denote the Jacobson radical lying in $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$.

For any $\alpha \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, set the α**-radical functor** $F_\alpha := \mathrm{rad} \circ \alpha$.

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For any $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, we define the (α, β)**-layer length** $\ell\ell_\alpha^\beta : \mathcal{C} \longrightarrow \mathbb{N} \cup \{\infty\}$ via $\ell\ell_\alpha^\beta(M) = \min\{i \geqslant 0 \mid \alpha \circ \beta^i(M) = 0\}$; and the α**-radical layer length** $\ell\ell^\alpha := \ell\ell_\alpha^{F_\alpha}$.

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Definition 2.2. (Huard-Lanzilotta-Mendoza Hernández, 2013)

For any $\alpha, \beta \in \mathrm{End}_{\mathbb{Z}}(\mathcal{C})$, we define the (α, β)**-layer length** $\ell\ell_\alpha^\beta : \mathcal{C} \longrightarrow \mathbb{N} \cup \{\infty\}$ via $\ell\ell_\alpha^\beta(M) = \min\{i \geqslant 0 \mid \alpha \circ \beta^i(M) = 0\}$; and the α**-radical layer length** $\ell\ell^\alpha := \ell\ell_\alpha^{F_\alpha}$.

Recall that a **torsion pair** (or **torsion theory**) for $\mathcal{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ of objects in $\mathcal{C}$ satisfying the following conditions.

(1) $\mathrm{Hom}_{\mathcal{C}}(M, N) = 0$ for any $M \in \mathcal{T}$ and $N \in \mathcal{F}$;

(2) an object $X \in \mathcal{C}$ is in $\mathcal{T}$ if $\mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{F}} = 0$;

(3) an object $Y \in \mathcal{C}$ is in $\mathcal{F}$ if $\mathrm{Hom}_{\mathcal{C}}(-, Y)|_{\mathcal{T}} = 0$.

Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair for $\mathcal{C}$. Recall that the **torsion radical** t attached to $(\mathcal{T}, \mathcal{F})$ is a functor in $\mathrm{End}_{\mathbb{Z}}(\mathcal{C})$ such that for any $M \in \mathcal{C}$,

$$0 \to t(M) \to M \to M/t(M) \to 0$$

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Relations with some homological invariants

3. Relations with some homological invariants

In this section, $\mathcal{A}$ is an abelian category.

Definition 3.1. (Iyama, 2003; Oppermann, 2009)

Let $M \in \mathcal{A}$. The **weak M-resolution dimension** of an object X in $\mathcal{A}$, denoted by $M\text{-w.resol.dim}\, X$, is defined as $\inf\{i \geq 0 \mid$ there exists an exact sequence

$$0 \longrightarrow M_i \longrightarrow M_{i-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow X \longrightarrow 0$$

in $\mathcal{A}$ with all M_j in $\operatorname{add} M\}$. The **weak M-resolution dimension** of $\mathcal{A}$, $M\text{-w.resol.dim}\, \mathcal{A}$, is defined as $\sup\{M\text{-w.resol.dim}\, X \mid X \in \mathcal{A}\}$. The **weak resolution dimension** of $\mathcal{A}$ is denoted by $\text{w.resol.dim}\, \mathcal{A}$ and defined as $\inf\{M\text{-w.resol.dim}\, \mathcal{A} \mid M \in \mathcal{A}\}$.

For a subclass $\mathcal{X}$ of $\mathcal{A}$, recall that a sequence $\mathbb{S}$ in $\mathcal{A}$ is called $\mathrm{Hom}_{\mathcal{A}}(\mathcal{X}, -)$**-exact** (resp. $\mathrm{Hom}_{\mathcal{A}}(-, \mathcal{X})$**-exact**) if $\mathrm{Hom}_{\mathcal{A}}(X, \mathbb{S})$ (resp. $\mathrm{Hom}_{\mathcal{A}}(\mathbb{S}, X)$) is exact for any $X \in \mathcal{X}$.

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3.1. Representation and global dimensions

Definition 3.1. (Auslander, 1971; Rouquier, 2006)

The **representation dimension** $\operatorname{rep.dim} \mathcal{A}$ of $\mathcal{A}$ is the smallest integer $i \geq 2$ such that for any $X \in \mathcal{A}$, there exists $M \in \mathcal{A}$ satisfying the following conditions.

(1) there exists a $\operatorname{Hom}_{\mathcal{A}}(\operatorname{add} M, -)$-exact exact sequence

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We call $A \in \mathcal{A}$ an **additive generating object** if $\operatorname{add} A$ is a generator for $\mathcal{A}$. It is trivial that if $A \in \mathcal{A}$ is an additive generating object, then all projective objects in $\mathcal{A}$ are in $\operatorname{add} A$.

Theorem 3.2.

Assume that $\mathcal{A}$ admits an additive generating object A. If $\mathcal{A}$ has enough projective objects and enough injective objects, then

$$\text{w.resol.dim}\,\mathcal{A} = \dim \mathcal{A} \leq \text{rep.dim}\,\mathcal{A} - 2.$$

As applications of Theorem 3.2, we get the following four corollaries.

Corollary 3.3.

If Λ is a right Morita ring, then

$$\text{w.resol.dim}\,\text{mod}\,\Lambda = \dim \text{mod}\,\Lambda \leq \min\{\text{r.gl.dim}\,\Lambda, \text{rep.dim}\,\Lambda - 2\}.$$

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From now on, Λ is an artin algebra.

- Λ is called n-**Gorenstein** if its left and right self-injective dimensions are at most n.
- Let $\mathcal{P}$ be the subcategory of $\mathrm{mod}\,\Lambda$ consisting of projective modules. A module $G \in \mathrm{mod}\,\Lambda$ is called **Gorenstein projective** if there exists a $\mathrm{Hom}_{\Lambda}(-,\mathcal{P})$-exact exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

in $\mathrm{mod}\,\Lambda$ with all P_i, P^i in $\mathcal{P}$ such that $G \cong \mathrm{Im}(P_0 \to P^0)$.
- Λ is said to be of **finite Cohen-Macaulay type** (**finite CM-type** for short) if there are only finitely many non-isomorphic indecomposable Gorenstein projective modules in $\mathrm{mod}\,\Lambda$.

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Corollary 3.4.

If Λ is an n-Gorenstein algebra of finite CM-type, then $\dim \operatorname{mod} \Lambda \leq n$.

For small $\dim \operatorname{mod} \Lambda$, we have the following

Corollary 3.5.

(1) (Beligiannis, 2008) $\operatorname{rep.dim} \Lambda \leq 2$ if and only if $\dim \operatorname{mod} \Lambda = 0$;

(2) if $\operatorname{rep.dim} \Lambda = 3$, then $\dim \operatorname{mod} \Lambda = 1$.

For a field k and $n \geq 1$, $\wedge(k^n)$ is the exterior algebra of k^n.

Corollary 3.6.

$\dim \operatorname{mod} \wedge(k^n) = n - 1$ for any $n \geq 1$.

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3.2. Finitistic dimension

- $\Omega^n(M)$: the n-th syzygy of M in $\operatorname{mod} R$.
- $\Omega^n(\mathcal{X}) := \{\Omega^n(M) \mid M \in \mathcal{X}\}$ for a subclass $\mathcal{X}$ of $\operatorname{mod} \Lambda$.
- $\mathcal{P}^{<\infty} := \{M \in \operatorname{mod} \Lambda \mid \operatorname{pd} M < \infty\}$.

Recall that the **finitistic dimension** $\operatorname{fin.dim} \Lambda$ of Λ is defined as $\sup\{\operatorname{pd} M \mid M \in \mathcal{P}^{<\infty}\}$. It is an unsolved conjecture that $\operatorname{fin.dim} \Lambda < \infty$ for every artin algebra Λ.

Proposition 3.7.

The following statements are equivalent.

(1) $\operatorname{fin.dim} \Lambda < \infty$;

(2) there exists some $n \geq 0$ such that $\operatorname{size}_{\operatorname{mod} \Lambda} \Omega^n(\mathcal{P}^{<\infty}) \leq 1$.

Corollary 3.8.

If $\dim \operatorname{mod} \Lambda \leq 1$, then $\operatorname{fin.dim} \Lambda < \infty$.

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Recall that the **finitistic dimension** $\operatorname{fin.dim} \Lambda$ of Λ is defined as $\sup\{\operatorname{pd} M \mid M \in \mathcal{P}^{<\infty}\}$. It is an unsolved conjecture that $\operatorname{fin.dim} \Lambda < \infty$ for every artin algebra Λ.

Proposition 3.7.

The following statements are equivalent.

(1) $\operatorname{fin.dim} \Lambda < \infty$;

(2) there exists some $n \geq 0$ such that $\operatorname{size}_{\operatorname{mod} \Lambda} \Omega^n(\mathcal{P}^{<\infty}) \leq 1$.

Corollary 3.8.

If $\dim \operatorname{mod} \Lambda \leq 1$, then $\operatorname{fin.dim} \Lambda < \infty$.

3.2. Finitistic dimension

- $\Omega^n(M)$: the n-th syzygy of M in $\operatorname{mod} R$.
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3.3. Igusa-Todorov algebras

Definition 3.8. (Wei, 2009)

For an integer $n \geq 0$, Λ is called n-**Igusa-Todorov** if there exists $V \in \operatorname{mod}\Lambda$ such that for any $M \in \operatorname{mod}\Lambda$, there exists an exact sequence $0 \longrightarrow V_1 \longrightarrow V_0 \longrightarrow \Omega^n(M) \oplus P \longrightarrow 0$ in $\operatorname{mod}\Lambda$ with $V_1, V_0 \in \operatorname{add} V$ and P projective.

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For any $n \geq 0$, the following statements are equivalent.

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The first assertion in the following proposition means that $\dim \operatorname{mod} \Lambda$ is an invariant for measuring how far Λ is from being 0-Igusa-Todorov.

Corollary 3.10.

(1) Λ is 0-Igusa-Todorov if and only if $\dim \operatorname{mod} \Lambda \leq 1$;

(2) if Λ is n-Igusa-Todorov, then $\dim \operatorname{mod} \Lambda \leq n+1$.

Moreover, we have the following

Corollary 3.11.

$\dim \operatorname{mod} \Lambda \leq 2$ if Λ is in one class of the following algebras.

(1) monomial algebras;

(2) left serial algebras;

(3) $\operatorname{rad}^{2n+1} \Lambda = 0$ and $\Lambda / \operatorname{rad}^n \Lambda$ is representation finite;

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3.4. $t_{\mathcal{S}}$-radical layer length

Let Λ be an artin algebra. Then $\operatorname{mod}\Lambda$ is a length-category.

- For a subclass $\mathcal{B}$ of $\operatorname{mod}\Lambda$, the **projective dimension** $\operatorname{pd}\mathcal{B}$ of $\mathcal{B}$ is defined as

$$\operatorname{pd}\mathcal{B} = \begin{cases} \sup\{\operatorname{pd}M \mid M \in \mathcal{B}\}, & \text{if } \mathcal{B} \neq \varnothing; \\ -1, & \text{if } \mathcal{B} = \varnothing. \end{cases}$$

- $\mathcal{S}^{<\infty}$: all pairwise non-isomorphism simple Λ-modules with finite projective dimension.
- $\mathcal{S}$: a subset of $\mathcal{S}^{<\infty}$.
- $\mathcal{S}'$: the set of all the others simple modules in $\operatorname{mod}\Lambda$.

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- $\mathfrak{F}(\mathcal{S}) := \{M \in \operatorname{mod}\Lambda \mid$ there exists a finite chain

$$0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$$

of submodules of M such that each quotient M_i/M_{i-1} is isomorphic to some module in $\mathcal{S}\}$.

- $\mathcal{T}_{\mathcal{S}} := \{M \in \operatorname{mod}\Lambda \mid \operatorname{top} M \in \operatorname{add}\mathcal{S}'\}$.

Then $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}))$ is a torsion pair. We denote the torsion radical attached to $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S}))$ by $t_{\mathcal{S}}$. Then $t_{\mathcal{S}}(M) \in \mathcal{T}_{\mathcal{S}}$ and $M/t_{\mathcal{S}}(M) \in \mathfrak{F}(\mathcal{S})$ for any $M \in \operatorname{mod}\Lambda$, and

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Let $\mathcal{S}$ be a subset of $\mathcal{S}^{<\infty}$. Then

$$\dim \operatorname{mod} \Lambda \leq \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda).$$

As a consequence, we have the following

Corollary 3.13.

(1) (Beligiannis, 2008) $\dim \operatorname{mod} \Lambda \leq \operatorname{LL}(\Lambda) - 1$;

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Proof.

(1) Let $\mathcal{S} = \varnothing$. Then $\operatorname{pd} \mathcal{S} = -1$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (\operatorname{mod} \Lambda, 0)$. According to Huard-Lanzilotta-Mendoza Hernández (2013), we have $t_{\mathcal{S}}(\Lambda) = \Lambda$ and $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \operatorname{LL}(\Lambda)$. It follows from Theorem 3.12 that $\dim \operatorname{mod} \Lambda \leq \operatorname{LL}(\Lambda) - 1$.

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Theorem 3.12.

Let $\mathcal{S}$ be a subset of $\mathcal{S}^{<\infty}$. Then

$$\dim \operatorname{mod} \Lambda \leq \operatorname{pd} \mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda).$$

As a consequence, we have the following

Corollary 3.13.

(1) (Beligiannis, 2008) $\dim \operatorname{mod} \Lambda \leq \mathrm{LL}(\Lambda) - 1$;

(2) (cf. Corollary 3.3 and Iyama, 2013) $\dim \operatorname{mod} \Lambda \leq \operatorname{gl.dim} \Lambda$.

Proof.

(1) Let $\mathcal{S} = \varnothing$. Then $\operatorname{pd} \mathcal{S} = -1$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (\operatorname{mod} \Lambda, 0)$. According to Huard-Lanzilotta-Mendoza Hernández (2013), we have $t_{\mathcal{S}}(\Lambda) = \Lambda$ and $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = \mathrm{LL}(\Lambda)$. It follows from Theorem 3.12 that $\dim \operatorname{mod} \Lambda \leq \mathrm{LL}(\Lambda) - 1$.

(2) Let $\mathcal{S} = \mathcal{S}^{<\infty} = \{$all simple modules in $\operatorname{mod} \Lambda\}$. Then $\operatorname{pd} \mathcal{S} = \operatorname{gl.dim} \Lambda$ and the torsion pair $(\mathcal{T}_{\mathcal{S}}, \mathfrak{F}(\mathcal{S})) = (0, \operatorname{mod} \Lambda)$. According to Huard-Lanzilotta-Mendoza Hernández (2013), we have $t_{\mathcal{S}}(\Lambda) = 0$ and $\ell\ell^{t_{\mathcal{S}}}(\Lambda) = 0$. It follows from Theorem 3.12 that $\dim \operatorname{mod} \Lambda \leq \operatorname{gl.dim} \Lambda$.

By choosing some suitable $\mathcal{S}$ and applying Theorem 3.12, we may obtain more precise upper bounds for $\dim \operatorname{mod} \Lambda$ than that in Corollary 3.13.

Example 3.14.

Consider the bound quiver algebra $\Lambda = kQ/I$, where k is a field and Q is given by

$$\begin{tikzcd}
 & 2n+1 & & & & \\
2n & 1 \arrow[l, "\alpha_{2n}"'] \arrow[u, "\alpha_{2n+1}"] \arrow[r, "\alpha_1"] \arrow[d, "\alpha_{n+1}"] & 2 \arrow[r, "\alpha_2"] & 3 \arrow[r, "\alpha_3"] & \cdots \arrow[r, "\alpha_{n-1}"] & n \\
 & n+1 \arrow[r, "\alpha_{n+2}"] & n+2 \arrow[r, "\alpha_{n+3}"] & n+3 \arrow[r, "\alpha_{n+4}"] & \cdots \arrow[r, "\alpha_{2n-1}"] & 2n-1
\end{tikzcd}$$

and I is generated by $\{\alpha_i\alpha_{i+1} \mid n+1 \leq i \leq 2n-2\}$ with $n \geq 5$.

By choosing some suitable $\mathcal{S}$ and applying Theorem 3.12, we may obtain more precise upper bounds for $\dim \operatorname{mod} \Lambda$ than that in Corollary 3.13.

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2n & \xleftarrow[\alpha_{2n}]{} & 1 & \xrightarrow{\alpha_1} & 2 & \xrightarrow{\alpha_2} & 3 & \xrightarrow{\alpha_3} & \cdots & \xrightarrow{\alpha_{n-1}} & n \\
 & & \downarrow{\scriptstyle \alpha_{n+1}} & & & & & & & & \\
 & & n+1 & \xrightarrow{\alpha_{n+2}} & n+2 & \xrightarrow{\alpha_{n+3}} & n+3 & \xrightarrow{\alpha_{n+4}} & \cdots & \xrightarrow{\alpha_{2n-1}} & 2n-1
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Example 3.14. (continued)

(1) We have $\mathrm{LL}(\Lambda) = n$ and $\mathrm{gl.dim}\,\Lambda = n - 1$. So by Corollary 3.13, we have $\dim \mathrm{mod}\,\Lambda \leq \min\{\mathrm{gl.dim}\,\Lambda, \mathrm{LL}(\Lambda) - 1\} = n - 1$.

(2) We have $\mathcal{S}^{<\infty} = \{$all simple modules in $\mathrm{mod}\,\Lambda\}$. Choose

$$\mathcal{S} = \{S(i) \mid 2 \leq i \leq n\}.$$

By computation, we have

$$\mathrm{pd}\,\mathcal{S} = 1 \text{ and } \ell\ell^{t_{\mathcal{S}}}(\Lambda) = \max\{\ell\ell^{t_{\mathcal{S}}}(P(i)) \mid 1 \leq i \leq 2n+1\} = 2.$$

Then by Theorem 3.12, we have

$$\dim \mathrm{mod}\,\Lambda \leq \mathrm{pd}\,\mathcal{S} + \ell\ell^{t_{\mathcal{S}}}(\Lambda) = 1 + 2 = 3.$$

The upper bound here is better than that in (1) since $n \geq 5$.

Example 3.14. (continued)

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(2) We have $\mathcal{S}^{<\infty} = \{$all simple modules in $\bmod \Lambda\}$. Choose

$$\mathcal{S} = \{S(i) \mid 2 \leq i \leq n\}.$$

By computation, we have

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Thank you!