On the 2-test modules for projectivity and weakly m-full ideals

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R: commutative Noetherian local ring

$\mathfrak{m}$: the unique maximal ideal of R

$k := R/\mathfrak{m}$: the residue class field of R

$\operatorname{mod} R$: the category of finitely generated R-modules

Definition

(1) $M \in \operatorname{mod} R$: n-test module for projectivity

$\underset{\text{def}}{\Longleftrightarrow}$ $X \in \operatorname{mod} R$, $\operatorname{Ext}_R^{1 \sim n}(X, M) = 0 \longrightarrow X$: projective

(2) $M \in \operatorname{mod} R$: n-Tor-test module for projectivity

$\underset{\text{def}}{\Longleftrightarrow}$ $X \in \operatorname{mod} R$: $\operatorname{Tor}^R_{1 \sim n}(X, M) = 0 \longrightarrow X$: projective

(3) $I \subsetneq R$: weakly $\mathfrak{m}$-full

$\underset{\text{def}}{\Longleftrightarrow}$ $I = \mathfrak{m}I :_R \mathfrak{m}$

Notation

- $M \in \operatorname{mod} R$

  $\operatorname{depth}_R M := \inf\{ i \geqq 0 \mid \operatorname{Ext}_R^i(k, M) \neq 0 \}$

- $M \in \operatorname{mod} R$

  $\Omega_R M$: a 1-st syzygy of M

  $\Omega_R^n M := \Omega_R(\Omega_R^{n-1} M) \quad (\forall n \geqq 2)$

  : an n-th syzygy of M

Example

(1) k, $\mathfrak{m}$: 1-test module for projectivity

therefore n-test module for projectivity

($\forall n \geq 1$),

(2) k : 1-Tor-test module for projectivity

($\therefore$ n-Tor-test module for projectivity ($\forall n \geq 1$))

(3) I : integrally closed ideal
(e.g. radical ideal)

or

$\mathrm{depth}_R(R/I) > 0$

$\Longrightarrow$ I : weakly $\mathfrak{m}$-full

Remark

(1) M : n-test module for projectivity

$\Longrightarrow \operatorname{depth}_R M \leqq n$

(2) $x \in \mathfrak{m}$: Non-zero divisor on M

- M : n-test module for projectivity

$\Longrightarrow M/xM$: n-test R-module for projectivity

- M/xM : n-test R-module for projectivity

$\Longrightarrow M$: $(n+1)$-test module for projectivity

- M : n-Tor-test module for projectivity

$\Longrightarrow M/xM$: n-Tor-test R-module for projectivity

(3) M : n-test module for projectivity

$\Longrightarrow \Omega_R M$: $(n+2)$-test module for projectivity

Theorem A

M : n-Tor-test module for projectivity

$\Rightarrow M, \Omega_R M, \Omega_R^2 M, \cdots, \Omega_R^n M$: n-test module for projectivity

Theorem B

Assume that I : weakly $\mathfrak{m}$-full and $\mathrm{Tor}_1(M, R/I) = 0$. Then, a free covering of M $0 \to N \to F \to M \to 0$ induces an exact sequence $0 \to N/IN \to F/IF \to M/IM \to 0$ such that $\mathrm{depth}_R(N/IN) > 0$.

Moreover, if I is $\mathfrak{m}$-primary then M is free.

Theorem C

I : weakly $\mathfrak{m}$-full, $\operatorname{depth}_R(R/I) = 0$

$\operatorname{Tor}_n^R(M, R/I) = 0$ & $\operatorname{depth}_R(\operatorname{Tor}_{n-1}^R(M, R/I)) > 0$

$\Longrightarrow \operatorname{proj.dim}_R M < n-1$

Corollary [Celikbas-Goto-Takahashi-Taniguchi] + [Iima]

I : $\mathfrak{m}$-primary weakly $\mathfrak{m}$-full ideal

$\Longrightarrow$ • R/I : 1-Tor-test module for projectivity

• R/I, I : 1-test module for projectivity

<u>Corollary</u> [Celikbas-Iima-Sadeghi-Takahashi]

I : weakly $\mathfrak{m}$-full & $\operatorname{depth}_R(R/I)=0$

$\Rightarrow$ • R/I : 2-Tor-test module for projectivity

• R/I, I, $\Omega^2_R(R/I)$: 2-test module for projectivity

References

[Celikbas-Goto-Takahashi-Taniguchi]
On the ideal case of a conjecture of Huneke and Wiegand, Proc. Edinb. Math Soc. (2) (to appear)

[Celikbas-Iima-Sadeghi-Takahashi]
On the ideal case of a conjecture of Auslander and Reiten, Bull. des Sciences Mathématiques 142 (2018), 94-107.

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