Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations

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AIM

- Introduce Cohen-Macaulay (CM) modules over Gorenstein differential graded (dg) algebras.
- Study the representation theory of CM dg modules.
- Classify representation finite d-self-injective dg algebras in terms of AR quiver.

Setting

Throughout k is a field and A is a differential graded (dg) k-algebra.
$D = \mathrm{Hom}_k(?, k)$ the graded k-dual.

- $\mathcal{D}(A)$ = derived category of A.
- $\mathcal{D}^{\mathrm{b}}(A) = \{M \in \mathcal{D}(A) \mid \dim_k \bigoplus_{i \in \mathbb{Z}} \mathrm{H}^i(A) < \infty\} \subset \mathcal{D}(A)$.
- $\mathrm{per}\, A$ = the thick subcategory of $\mathcal{D}(A)$ generated by A.

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Assumption

(1) A is *non-positive*, *i.e.* $A^i = 0$ for $i > 0$;

(2) A is *proper*, *i.e.* $A \in \mathcal{D}^{\mathrm{b}}(A)$;

(3) A is *Gorenstein*, *i.e.* $\mathrm{per}\, A$ coincides with $\mathrm{thick}(DA)$, the thick subcategory generated by DA.

In this talk, we always assume A satisfies the assumption above.

CM dg modules

Definition

(1) A dg A-module $M \in \mathcal{D}^{\mathrm{b}}(A)$ is called *Cohen-Macaulay* if $\mathrm{H}^i(M) = 0$ and $\mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(A)}(M, A[i]) = 0$ for $i > 0$;

(2) $\mathrm{CM}\, A := \{\text{Cohen-Macaulay dg } A\text{-modules}\} \subset \mathcal{D}^{\mathrm{b}}(A)$.

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- If A is a finite-dimensional Gorenstein k-algebra. Then

$$\mathrm{CM}\, A = \{M \in \mathrm{mod}\, A \mid \mathrm{Ext}_A^{>0}(M, A) = 0\}.$$

- A is *d-self-injective* (resp. d-symmetric) if DA is isomorphic to $A[-d]$ in $\mathcal{D}(A)$ (resp. $\mathcal{D}(A^{\mathrm{e}})$), $d \geq 0$. In this case,

$$\mathrm{CM}\, A = \{M \in \mathcal{D}^{\mathrm{b}}(A) \mid \mathrm{H}^i(M) = 0 \text{ for } i > 0 \text{ and } i < -d\}.$$

Properties of CM dg modules

Theorem

Let A be a non-positive proper Gorenstein dg algebra. Then

(1) $\mathrm{CM}\,A$ *is a Frobenius extriangulated category with* $\mathrm{Proj}(\mathrm{CM}\,A) = \mathrm{add}\,A$ *(in the sense of [Nakaoka-Palu]);*

(2) *The stable category* $\underline{\mathrm{CM}}A := (\mathrm{CM}\,A)/[\mathrm{add}\,A]$ *is a triangulated category;*

(3) *The composition* $\mathrm{CM}\,A \hookrightarrow \mathcal{D}^{\mathrm{b}}(A) \to \mathcal{D}^{\mathrm{b}}(A)/\,\mathrm{per}\,A$ *induces a triangle equivalence*

$$\underline{\mathrm{CM}}A = (\mathrm{CM}\,A)/[\mathrm{add}\,A] \simeq \mathcal{D}^{\mathrm{b}}(A)/\,\mathrm{per}\,A = \mathcal{D}_{\mathrm{sg}}(A);$$

(4) $\underline{\mathrm{CM}}A$ *admits a Serre functor and* $\mathrm{CM}\,A$ *admits almost split extensions;*

(5) $\mathrm{CM}\,A$ *is functorially finite in* $\mathcal{D}^{\mathrm{b}}(A)$.

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- Δ : Dynkin diagram. Let $A = k\Delta \oplus D(k\Delta)[d]$, $d \geq 0$, be the trivial extension dg k-algebra with 0 differential. Then A is d-symmetric, and moreover, $\underline{\mathrm{CM}}A \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, k\Delta)/\nu[d+1]$.

Q: $\begin{tikzcd} 1 \arrow[r, bend right, "\alpha_1"'] & 2 \arrow[l, bend right, "\alpha_2"'] \arrow[r, bend left, "\beta_2"] & 3 \arrow[l, bend left, "\beta_1"] \end{tikzcd}$, $I = < \alpha^3, \alpha^2 - \beta^2, \beta^3, \alpha\beta, \beta\alpha >$.

$A := kQ/I$ dg algebra with $\deg \alpha_1 = \deg \beta_1 = 0$, $\deg \alpha_2 = \deg \beta_2 = -1$ and 0 differential. $\Gamma(\underline{\mathrm{CM}}A) = \mathbb{Z}A_3/\mathbb{S}[2]$:

$$\begin{array}{ccccccccccccc}
 & \begin{smallmatrix}2\\3'\end{smallmatrix} & & \begin{smallmatrix}3\\2\end{smallmatrix} & & 1 & & \begin{smallmatrix}1'\\2'\end{smallmatrix} & & 3' & & \begin{smallmatrix}2\\1'\end{smallmatrix} & \\
\cdots & & 2 & & \begin{smallmatrix}1\ 3\\2\end{smallmatrix} & & 2' & & \begin{smallmatrix}1'\ 3'\\2'\end{smallmatrix} & & \begin{smallmatrix}2\\1'\ 3'\end{smallmatrix} & & \cdots \\
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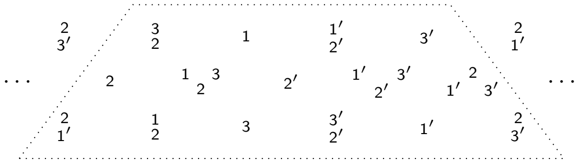

Example 1

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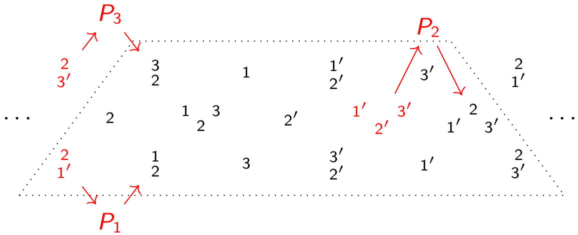

Example 2

Theorem

Let $A := k[X]/(X^{n+1}), n \geq 1$, be the truncated polynomial dg k-algebra with $\deg X = -d$ for $d \geq 0$ and 0 differential. Then

$$\underline{\mathrm{CM}}A \simeq \mathcal{C}_{d+1}(A_n) := \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, kA_n)/\nu[-d-1].$$

Let $A_i := k[X]/(X^i)$, $i = 1, 2, \cdots, n$ be the dg A-modules and $T_{i,t} = A_i[td]$. If d is even. Let $N := \frac{(n+1)d+2}{2}$. Then $\Gamma(\mathrm{CM}\, A)$:

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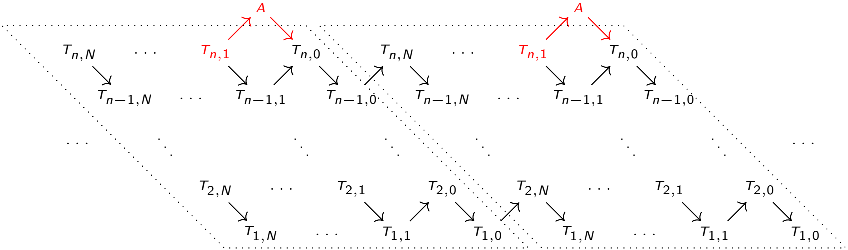

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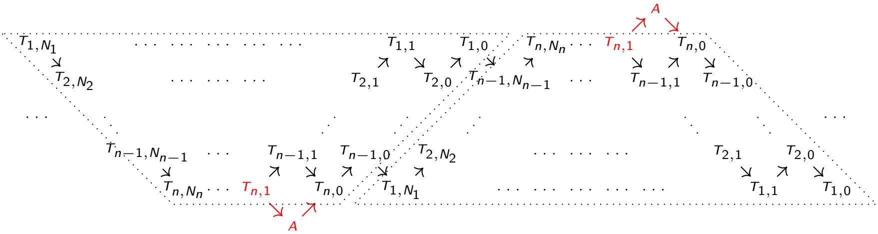

$(-d-1)$-CY configuration

- T: k-linear Hom-finite Krull-Schmidt triangulated category.
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Definition

We call C a $(-d-1)$-*Calabi-Yau configuration* (or $(-d-1)$-*CY configuration*) of $\mathcal{T}$, if the following conditions hold.

(1) $\dim_k \mathrm{Hom}_{\mathcal{T}}(X, Y) = \delta_{X,Y}$ for $X, Y \in C$;

(2) $\mathrm{Hom}_{\mathcal{T}}(X, Y[-j]) = 0$ for any two objects X, Y in C and $0 < j \leq d$;

(3) For any indecomposable object M in $\mathcal{T}$, there exists $X \in C$ and $0 \leq j \leq d$, such that $\mathrm{Hom}_{\mathcal{T}}(X, M[-j]) \neq 0$.

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Theorem

Let C be a $(-d-1)$-CY configuration of $\mathcal{T}$. If $\mathcal{T}$ has a Serre functor $\mathbb{S}$, then $\mathbb{S}C[d+1] = C$.

Simple dg modules

$$\begin{tikzcd}
A: \arrow[d] & \cdots \arrow[r] & A^{-2} \arrow[r, "d^{-2}"] & A^{-1} \arrow[r, "d^{-1}"] & A^0 \arrow[r] \arrow[d] & 0 \arrow[r] & \cdots \\
\mathrm{H}^0(A): & & & & \mathrm{H}^0(A) & &
\end{tikzcd}$$

We may regard $\mathrm{H}^0(A)$-modules as dg A-modules.
$\mathcal{S}_{\mathcal{A}} = \{\text{simple dg } A\text{-modules}\} := \{\text{simple } \mathrm{H}^0(A)\text{-modules (in degree 0)}\}$.

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Theorem

Let A be a d-self-injective dg k-algebra. Then $\mathcal{S}_{\mathcal{A}}$ is a $(-d-1)$-CY configuration of $\underline{\mathrm{CM}}A$.

Main result

Theorem

Let C be a subset of vertices of $\mathbb{Z}\Delta/\mathbb{S}[d+1]$. TFAE

(1) *C is a $(-d-1)$-CY configuration;*

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- $(\mathbb{Z}\Delta)_C$: a new translation quiver by adding to $\mathbb{Z}\Delta$ a vertex P_X and two arrows $X \to P_X \to \tau^{-1}X$ for each $X \in C$.

Sketch of proof.

- $B = k\Delta \oplus D(k\Delta)[d]$, Δ with an alternating orientation.

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- The functor $\mathcal{D}^{\mathrm{b}}(B) \to \mathcal{D}_{\mathsf{sg}}(B) \cong \underline{\mathsf{CM}}B$ induces a surjective map

$$\begin{array}{ccc} \{\text{SMCs in } \mathcal{D}^{\mathrm{b}}(B)\} & \twoheadrightarrow & \{(-d-1)\text{-CY configurations in } \underline{\mathsf{CM}}B\} \\ \mathcal{S} & \longmapsto & C \end{array}$$

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- $A = \mathscr{E}\mathrm{nd}_B(P)$ is what we want.

$\square$

References

- Haibo Jin, *Cohen-Macaulay differential graded modules and negative Calabi-Yau configurations*, arXiv:1812.03737.
- Haibo Jin, *Simple-minded reductions of triangulated categories*, arXiv:1907.05114.
- Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) 27 (1994), no. 1, 63–102.
- Bernhard Keller, *On triangulated orbit categories*. Doc. Math. 10 (2005), 551–581.
- Steffen Koenig, Dong Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-dimensional algebras*, Doc. Math. 19 (2014), 403–438.

References

Hiroyuki Nakaoka, Yann Palu, *Mutation via Hovey twin cotorsion pairs and model structures in extriangulated categories*, arXiv:1605.05607.

Christine Riedtmann, *Representation-finite self-injective algebras of class A_n*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), pp. 449–520, Lecture Notes in Math., 832, Springer, Berlin, 1980.

Christine Riedtmann, *Representation-finite self-injective algebras of class D_n*. Compositio Math. 49 (1983), no. 2, 231–282.