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Rudimentary rings: Rings have a faithful indecomposable endoregular module

(joint work with Cosmin Roman and Xiaoxiang Zhang)

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A right primitive ring R

$\exists$ a faithful simple right R-module M

Schur's Lemma

If M is a simple right R-module then $\mathrm{End}_R(M)$ is a division ring.

Lee-Rizvi-Roman

M is an indecomposable endoregular module
if and only if $\mathrm{End}_R(M)$ is a division ring.

From the above 3 observations, we consider:

A right rudimentary ring R

- $\exists$ a faithful right R-module M
- $\mathrm{End}_R(M)$: a division ring

Equivalently, a ring R which has
a faithful indecomposable endoregular right R-module M.

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Plan :

1. Historical Background for Primitive Rings.

2. Historical Background for Endoregular Modules.

3. Observations.

4. A generalization of primitive rings: Rudimentary rings.

5. Applications of indecomposable endoregular modules.

1. Historical background for primitive rings :

In 1908 (Wedderburn)

A simple artinian ring $R \cong$ an $n \times n$ matrix ring over a division ring for some $n \in \mathbb{N}$.

In 1927 (Wedderburn-Artin)

A semisimple artinian ring $R \Leftrightarrow$ a ring direct sum of a finite number of simple artinian rings.

In 1945 (Jacobson)

A right primitive ring $R \cong$ a dense subring of the endomorphism ring of a left vector space over a division ring.

Definition

A ring R is called right primitive if there exists a faithful simple right R-module.
A left primitive ring is defined similarly with left R-modules.

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Examples for primitive rings

- any simple ring;
- any full linear ring;

e.g., the endomorphism ring of any left vector space over a division ring;
- Weyl algebras over fields of characteristic zero;
- if M is a simple right R-module

then $R/\mathbf{r}_R(M)$ is a right primitive ring.

In 1961 (Posner)

A right primitive ring $R \Leftrightarrow$ a right primitive ring $\mathrm{Mat}_n(R)$.

In 1964 Bergman (Also, in 1988 Jategaonkar)

right primitive $\neq$ left primitive.

2. Historical background for endoregular modules :
Recall that a ring R is said to be von Neumann regular if for any $r \in R$, there exists $s \in R$ such that $r = rsr$.

In 1958 (L. Fuchs)

Fuchs raised the question of characterizing abelian groups whose endomorphism rings are von Neumann regular.

In 1967 (K.M. Rangaswamy)

Rangaswamy answered the question for groups.

In 1971 R. Ware (Also, in 1975 B. Stenström)

Ware and Stenström also answered the question for modules.

In 1948 (G. Azumaya independently)

The following conditions are equivalent for a right R-module M:
(a) $\mathrm{End}_R(M)$ is a von Neumann regular ring;
(b) $\mathrm{Ker}\,\varphi \leq^{\oplus} M$ and $\mathrm{Im}\,\varphi \leq^{\oplus} M$ for all $\varphi \in \mathrm{End}_R(M)$

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There is another general module theoretical setting of the notion of a von Neumman regular ring:

Definition

A module M is called endoregular
if its endomorphism ring is a von Neumann regular ring.

Example

- any semisimple module;
- any nonsingular (even $\mathcal{K}$-nonsingular) continuous (injective) module (thus, $\mathbb{Q}^{(\mathbb{R})}$ is an endoregular $\mathbb{Z}$-module);
- any finitely generated projective module over a von Neumann regular ring.

Theorem

The following conditions are equivalent for a module M:

(a) *M is an indecomposable endoregular module;*

(b) *$End_R(M)$ is a division ring.*

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Equivalently, a ring R which has
a faithful indecomposable endoregular right R-module M.

4. **Rudimentary rings** :

We introduce the generalized notion of primitive rings.

Definition

A ring R is called right rudimentary if there exists a faithful right R-module M such that $\mathrm{End}_R(M)$ is a division ring.
A left rudimentary ring is defined similarly with left R-modules.

Example

- any right primitive ring (hence any simple ring)
e.g., the endomorphism ring of any left vector space over a division ring;
- any right weakly primitive ring;
- any right Ore domain;
- any prime right Goldie ring;
- if M is an indecomposable endoregular right R-module then $R/\mathbf{r}_R(M)$ is a right rudimentary ring.

More examples of right rudimentary rings:

Theorem

Let Q be a right rudimentary ring with a faithful right Q-module M such that $End_Q(M)$ is a division ring.
If R is a right or left order in Q
then M_R is also faithful and $End_Q(M) = End_R(M)$.
Hence R is right rudimentary.

Corollary

(i) Every prime right Goldie ring (and hence every prime PI-ring) is right rudimentary.
(ii) Every right Ore domain is right rudimentary.

It is well known result for commutative primitive rings:

Proposition

A commutative ring R is a (right) primitive ring iff R is a field.

We generalize the above Proposition to rudimentary rings for the commutative case.

Proposition

A commutative ring R is rudimentary iff R is an integral domain.

Corollary

The center of a right rudimentary ring is a rudimentary ring.

Remark

There is no proper central idempotents in a right rudimentary ring.

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Next example shows that the right rudimentary property is not inherited by a corner ring, in general.

Example

The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 \\ 0 & 0 & \mathbb{Z} \end{pmatrix}$ is right rudimentary since $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ is a faithful right R-module and $\mathrm{End}_R(M) \cong \mathbb{Q}$.
Let $e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be the idempotent.
Then $eRe \cong \mathbb{Z} \times \mathbb{Z}$, which is not rudimentary.

In spite of the previous example,
the rudimentary property is a Morita invariant as shown next.
In fact, in the previous example, $ReR \neq R$.

Theorem

The rudimentary property is a Morita invariant.

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In fact, in the previous example, $ReR \neq R$.

Theorem

The rudimentary property is a Morita invariant.

A number of authors have studied rings R for which the Converse of Schur Lemma holds. Such a ring R is said to be a CSL ring. Every strongly regular ring is a CSL ring.

Next, we show:

Proposition

Every right rudimentary right CSL ring is right primitive.

5. Applications of indecomposable endoregular modules :

In 1949, T. Szele showed that there is no noncommutative division ring as the endomorphism ring of an abelian group.

In 1949 (T. Szele)

Let M be an abelian group such that $\mathrm{End}_{\mathbb{Z}}(M)$ is a division ring. Then M is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}_p$.

In 1970, Ware and Zelmanowitz extended Szele result that there is no noncommutative division ring as the endomorphism ring of an module over a commutative ring.

In 1970 (Ware and Zelmanowitz)

Let R be a commutative ring and let M be a right R-module. Then M is an indecomposable endoregular module iff M is R-isomorphic to $Q(R/P)$ where $P = r_R(M)$ and $Q(R/P)$ is the maximal ring of quotients of R/P.

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Observation

Let $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & \mathbb{Z} & \mathbb{Z} \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.

Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.

Then $\mathrm{End}_{R_1}(M) \cong \mathrm{End}_{R_2}(M) \cong \mathbb{Q}$.

An $n \times n$ partial matrix ring $\mathrm{PM}_n(A)$ over a ring A is a subring of a full $n \times n$ matrix ring over A, with elements matrices whose entries are either elements of A or are 0 such that nonzero entries are independent of each other.
That is, $\mathrm{PM}_n(A) = \sum_{(i,j)\in\mathcal{U}} e_{ij}A$ where e_{ij} are matrix units and $\mathcal{U}$ is a subset of the index set $\mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \dots, n\}$.

Theorem

Let A be a commutative ring and $R = \mathrm{PM}_n(A)$.
Let $M = \prod_{i=1}^{n} N_i$ be a direct product of right A-modules N_i.
If $\mathrm{End}_R(M)$ is a division ring
then $\mathrm{End}_R(M) \cong Q(A/P)$ for some prime ideal P of A.

Example

Let $R_3 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_4 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.
Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.
Then $\mathrm{End}_{R_3}(M) \cong \mathrm{End}_{R_4}(M) \cong \mathbb{Q}$.

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Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.
Then $\mathrm{End}_{R_3}(M) \cong \mathrm{End}_{R_4}(M) \cong \mathbb{Q}$.

Observation

Let $R_5 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ and $R_6 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.
Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.
Then $\text{End}_{R_5}(M) \cong \mathbb{Q}$ and $\text{End}_{R_6}(M) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Q} \times \mathbb{Q}$.

Lemma

Let M be a faithful right R-module, $S = End_R(M)$ be a field, and A be an integral domain such that $Q(A) = S$. If $dim_S(M) = n$ for some $n \in \mathbb{N}$ and the ring $R = \sum_{(i,j)\in\mathcal{U}} e_{ij}A$ where $\mathcal{U} \subseteq \mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$, then there exist the smallest cardinal index set $\mathcal{V}$ and a matrix ring $R' = \sum_{(i,j)\in\mathcal{V}} e_{ij}A$ such that $End_{R'}(M) \cong S$, $\mathcal{V} \subseteq \mathcal{U}$, and $|\mathcal{V}| = 2n - 1$.

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Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.
Then $\mathrm{End}_{R_5}(M) \cong \mathbb{Q}$ and $\mathrm{End}_{R_6}(M) \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbb{Q} \times \mathbb{Q}$.

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Let $R_7 = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_8 = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} & 0 & 0 \\ 0 & 0 & \mathbb{Z} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ be rings.
Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} & \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring.
Then $\mathrm{End}_{R_7}(M) \cong \mathrm{End}_{R_8}(M) \cong \mathbb{Q}$.

Lemma

Let M be a faithful right R-module and $S = End_R(M)$ be a field.
Consider $\mathcal{B} = \{A \mid S = Q(A), A \text{ is a ring}\}$
such that $\mathcal{B}$ is closed under the intersection.
Suppose $dim_S(M) = n$ for some $n \in \mathbb{N}$ and $R = \sum_{(i,j)\in\mathcal{U}} e_{ij}A_{ij}$
where $A_{ij} \in \mathcal{B}$ and $\mathcal{U} \subseteq \mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$,
then for $A = \bigcap_{(i,j)\in\mathcal{U}} A_{ij}$,
a ring $R' = \sum_{(i,j)\in\mathcal{U}} e_{ij}A$ satisfies $End_{R'}(M) = S$.

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a ring $R' = \sum_{(i,j)\in\mathcal{U}} e_{ij}A$ satisfies $End_{R'}(M) = S$.

Theorem

Let M be a faithful right R-module and $S = End_R(M)$ be a field. Consider $\mathcal{B} = \{A \mid S = Q(A), A \text{ is a ring}\}$ such that $\mathcal{B}$ is closed under the intersection. If $dim_S(M) = n$ for some $n \in \mathbb{N}$ and the ring $R = \sum_{(i,j)\in\mathcal{U}} e_{ij}A_{ij}$ where $A_{ij} \in \mathcal{B}$ and $\mathcal{U} \subseteq \mathcal{I} \times \mathcal{I}$, $\mathcal{I} = \{1, 2, \ldots, n\}$, then for $A = \cap_{(i,j)\in\mathcal{U}} A_{ij}$, there exist the smallest cardinal index set $\mathcal{V}$ and a matrix ring $R' = \sum_{(i,j)\in\mathcal{V}} e_{ij}A$ such that $End_{R'}(M) = S$, $\mathcal{V} \subseteq \mathcal{U}$, and $|\mathcal{V}| = 2n - 1$.

Example

Let $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $R_2 = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$,

$R_3 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}, R_4 = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}, R_5 = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}, R_6 = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{pmatrix}, R_7 = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix},$

$R_8 = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, R_9 = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}, R_{10} = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}, R_{11} = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{pmatrix}, R_{12} = \begin{pmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$

$R_{13} = \begin{pmatrix} \mathbb{Z} & m\mathbb{Z} \\ n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, $R_{14} = \begin{pmatrix} \mathbb{Z} & n\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, and $R_{15} = \begin{pmatrix} \mathbb{Z} & 0 \\ n\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.
be rings where $m, n \in \mathbb{Z}$.

Consider a right module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ over each ring. Then $\text{End}_{R_i}(M) \cong \mathbb{Q}$ where $1 \leq i \leq 15$.

The condition "$S = Q(A)$" in the set $\mathcal{B}$ in the previous theorem is not superfluous as the next example shows.

Example

Let $R = \begin{pmatrix} \mathbb{Z}[i] & \mathbb{Q}[i] \\ 0 & \mathbb{Z} \end{pmatrix}$ be the ring and $M = \begin{pmatrix} \mathbb{Q}[i] & \mathbb{Q}[i] \end{pmatrix}$ be the right R-module. Then $\mathrm{End}_R(M) \cong \mathbb{Q}[i]$.
However, $\mathrm{End}_R(M) \cong \mathbb{Q}[i] \neq Q(\mathbb{Z}) = \mathbb{Q}$.

Example

(i) Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ be the ring and $M = \begin{pmatrix} \mathbb{Q}[i] & \mathbb{Q}[i] \end{pmatrix}$ be the right R-module. Then $\mathrm{End}_R(M) \cong \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.
(ii) Let $R = \mathrm{Mat}_2(\mathbb{Z})$ be the ring and $M = \begin{pmatrix} \mathbb{Q}[i] & \mathbb{Q}[i] \end{pmatrix}$ be the right R-module. Then $\mathrm{End}_R(M) \cong \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ and $M \cong \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{pmatrix}$.
(iii) Let the ring $R = \begin{pmatrix} \mathbb{Q} & 0 & \mathbb{Q} & 0 \\ 0 & \mathbb{Z} & 0 & \mathbb{Z} \\ 0 & 0 & \mathbb{Q} & 0 \\ 0 & 0 & 0 & \mathbb{Z} \end{pmatrix}$ and the right R-module $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Z}_2 & \mathbb{Q} & \mathbb{Z}_2 \end{pmatrix}$. Then $\mathrm{End}_R(M) \cong \begin{pmatrix} \mathbb{Q} & 0 \\ 0 & \mathbb{Z}_2 \end{pmatrix}$ is semisimple artinian.

We can build a decreasing chain of right rudimentary rings, whose intersection is no longer right rudimentary.
The next example illustrates this statement.

Example

Let $R_k = \begin{pmatrix} \mathbb{Z} & 2^k\mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$ for any $k \in \mathbb{N}$.
Then $M = \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \end{pmatrix}$ has the property that $\mathrm{End}_{R_k}(M) \cong \mathbb{Q}$, $\forall\ k \in \mathbb{N}$.
However, $\mathrm{End}_{\bigcap R_k}(M) \cong \mathbb{Q} \times \mathbb{Q}$ is not a division ring.
Moreover, $\bigcap_k R_k = \begin{pmatrix} \mathbb{Z} & 0 \\ 0 & \mathbb{Z} \end{pmatrix}$ is not a right rudimentary ring.

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Bibliography

Thank you.

- L. Fuchs, Abelian groups, Publishing House of the Hungarian Academy of Sciences, Budapest (1958)
- N. Jacobson, The radical and semi-simplicity for arbitrary rings, Amer. J. Math., **1945** *67*, 300–320
- J. Lambek, Lectures on rings and modules, Blaisdell Publishing Co. Ginn and Co., Waltham, Mass.-Toronto(Ont.)-London (1966)
- G. Lee; S.T. Rizvi; C.S. Roman, Modules whose endomorphism rings are von Neumann regular, Comm. Algebra, **2013** *41(11)*, 4066–4088
- G. Lee; C.S. Roman; X. Zhang, Modules whose endomorphism rings are division rings, Comm. Algebra, **2014** *42(12)*, 5205–5223
- G. Marks; M. Schmidmeier, Extensions of simple modules and the converse of Schur's lemma, Trends in Mathematics, Birkhäuser/Springer Basel AG, Basel, **2010**, 229-237

E.C. Posner, Primitive matrix rings, Arch. Math., **1961** 12, 97–101

T. Szele, Die Abelschen Gruppen ohne eigentliche Endomorphismen (German), Acta Univ. Szeged. Sect. Sci. Math., **1949** *13*, 54–56

R. Ware; J. Zelmanowitz, Simple endomorphism rings, Amer. Math. Monthly, **1970** *77(9)*, 987–989

J.H.M. Wedderburn, On Hypercomplex Numbers, Proc. London Math. Soc., **1908** *S2-6(1)*, 77–118

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