AS-regluarity of geometric algebras of plane cubic curves

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AS-regular algebras

Throughout this talk,

- k: an algebraically closed field of characteristic 0.
- $A = \bigoplus_{i \in \mathbb{N}} A_i$: a connected graded k-algebra finitely generated in degree 1 (i.e. $A_0 = k$).

Definition 1 [Artin-Schelter, 1987]

A noetherian connected graded algebra A is called d-dimensional Artin-Schelter regular (AS-regular) algebra if

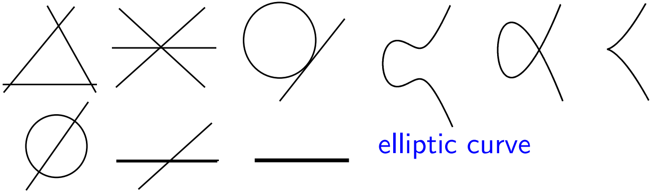
1. $\operatorname{gldim} A = d < \infty$.
2. $\operatorname{Ext}^i_A(k, A) = \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

3-dimensional quadratic AS-regular algebras

- If A is a 3-dimensional AS-regular algebra finitely generated in degree 1, then A is isomorphic to

$$k\langle x, y, z\rangle/(f_1, f_2, f_3) \text{ or } k\langle x, y\rangle/(g_1, g_2)$$

  where $f_1, f_2, f_3 \in k\langle x, y, z\rangle_2$ (quadratic) and $g_1, g_2 \in k\langle x, y\rangle_3$ (cubic) ([Artin-Schelter, 1987]).
- There exists a one-to-one correspondence between the set of 3-dimensional quadratic AS-regular algebras and a set of pairs (E, σ) where E is the projective plane $\mathbb{P}^2_k$ or a plane cubic curve and $\sigma \in \mathrm{Aut}_k E$ ([Artin-Tate-Van den Bergh, 1990]).



elliptic curve

Geometric algebras

- For a quadratic algebra $A = k\langle x_1, \cdots, x_n\rangle/I$ (i.e. I is generated by $I_2 \subset k\langle x_1, \ldots, x_n\rangle_2$),

$$\Gamma_A := \{(p, q) \in \mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1} \mid f(p, q) = 0, \forall f \in I_2\}.$$

Definition 2 [Mori, 2006]

$A = k\langle x_1, \ldots, x_n\rangle/I$: a quadratic algebra.

1. A satisfies (G1) ($\mathcal{P}(A) = (E, \sigma)$) $:\Longleftrightarrow$ there exists a pair (E, σ) (where $E \subset \mathbb{P}_k^{n-1}$ is a closed subscheme and $\sigma \in \mathrm{Aut}_k E$) such that $\Gamma_A = \{(p, \sigma(p)) \in \mathbb{P}_k^{n-1} \times \mathbb{P}_k^{n-1} \mid p \in E\}$.
2. A satisfies (G2) ($A = \mathcal{A}(E, \sigma)$) $:\Longleftrightarrow$ there exists a pair (E, σ) (where $E \subset \mathbb{P}_k^{n-1}$ is a closed subscheme and $\sigma \in \mathrm{Aut}_k E$) such that $I_2 = \{f \in k\langle x_1, \ldots, x_n\rangle_2 \mid f(p, \sigma(p)) = 0, \forall p \in E\}$.
3. A: *geometric* $:\Longleftrightarrow$ A satisfies (G1), (G2) and $A = \mathcal{A}(\mathcal{P}(A))$.

Example

Example

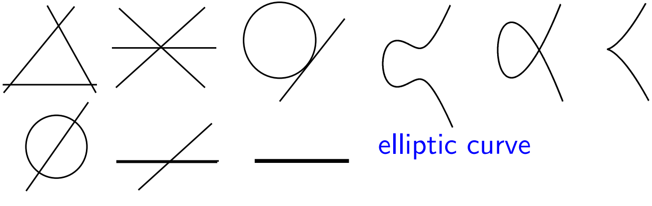
1. A commutative polynomial ring $A = k[x, y, z]$ is a geometric algebra with $\mathcal{P}(A) = (\mathbb{P}^2_k, \mathrm{id})$.
2. A 3-dimensional Sklyanin algebra

$$A = k\langle x, y, z\rangle/(ayz + bzy + cx^2, azx + bxz + cy^2, axy + byx + cz^2)$$

is a geometric algebra with $\mathcal{P}(A) = (E, \sigma_p)$ where E is an elliptic curve in $\mathbb{P}^2_k$ and σ_p is a translation by a point $p = (a : b : c) \in E$.

Geometric algebras of plane cubic curves

- Every 3-dimensional quadratic AS-regular algebra is a geometric algebra $\mathcal{A}(E, \sigma)$ where E is $\mathbb{P}^2_k$ or a palne cubic curve ([ATV]).



- In general, the converse is not true.
  - If $E \subset \mathbb{P}^2_k$ is singular, then a geometric algebra $A = \mathcal{A}(E, \sigma)$ is AS-regular for almost all $\sigma \in \mathrm{Aut}_k E$ ([Itaba-M, 2019]).
- In this talk, we mainly explain the case when $E \subset \mathbb{P}^2_k$ is non-singular, i.e., an elliptic curve.

Elliptic curve (Hesse form) · The j-invariant

Elliptic curve (Hesse form)

- We use a Hesse form
$$E = \mathcal{V}(f), f = x^3 + y^3 + z^3 - 3\lambda xyz \quad (\lambda \in k, \lambda^3 \neq 1).$$
  - An elliptic curve in $\mathbb{P}^2_k$ can be written by this form up to isomorphism.
- On an elliptic curve E in $\mathbb{P}^2_k$, we can define an addition with the zero element $0_E := (1 : -1 : 0) \in E$.
- For $p \in E$, an automorphism $\sigma_p \in \mathrm{Aut}_k E$ is defined by $\sigma_p(q) := p + q$ for $q \in E$, called a translation.

- The j-invariant of an elliptic curve is given by $j(E) = \frac{27\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3}$.
- $E \cong E'$ if and only if $j(E) = j(E')$.

Automorphism group

- $T := \{\sigma_p \in \mathrm{Aut}_k E \,|\, p \in E\} \le \mathrm{Aut}_k E$: the set of translations.
- $\mathrm{Aut}_k(E, 0_E) := \{\sigma \in \mathrm{Aut}_k E \,|\, \sigma(0_E) = 0_E\} \le \mathrm{Aut}_k E$.

Lemma 3 [Itaba-M, 2018]

$\mathrm{Aut}_k(E, 0_E) = \langle \tau \rangle$ where τ is given by

$$\begin{cases} \text{(i) } \tau(a:b:c) := (b:a:c), \ (\text{if } j(E) \neq 0, 12^3, \ |\tau| = 2), \\ \text{(ii) } \tau(a:b:c) := (b:a:\varepsilon c), \ (\text{if } j(E) = 0, \ |\tau| = 6), \\ \text{(iii) } \tau(a:b:c) := (\varepsilon^2 a + \varepsilon b + c : \varepsilon a + \varepsilon^2 b + c : a+b+c), \\ \quad (\text{if } j(E) = 12^3, \ |\tau| = 4), \end{cases}$$

for $(a:b:c) \in E$, where ε is a primitive 3rd root of unity.

$$\mathrm{Aut}_k E \cong T \rtimes \mathrm{Aut}_k(E, 0_E) = \{\sigma_p \tau^i \mid \sigma_p \in T, \ i \in \mathbb{Z}_{|\tau|}\}.$$

AS-regularity of geometric algebras

- $E[3] := \{p \in E \mid 3p = 0_E\}$: the set of 3-torsion points.

Lemma 4 [Itaba-M, 2018]

Let $A = \mathcal{A}(E, \sigma_p \tau^i)$ be a quadratic algebra satisfying the condition (G2). Then $A = \mathcal{A}(E, \sigma_p \tau^i)$: geometric algebra $\Longleftrightarrow p \in E \setminus E[3]$.

Theorem 5 [Itaba-M, 2019]

Let E be an elliptic curve in $\mathbb{P}^2_k$ and $A = \mathcal{A}(E, \sigma_p \tau^i)$ a geometric algebra where $p \in E \setminus E[3]$ and $i \in \mathbb{Z}_{|\tau|}$. Then A is a 3-dimensional quadratic AS-regular algebra if and only if $p - \tau^i(p) \in E[3]$.

If $i \neq 0$, then the number of points $p \in E$ which satisfy the condition $p - \tau^i(p) \in E[3]$ is finite.

Example 1

The case of $j(E) \neq 0, 12^3$

A generator $\tau \in \mathrm{Aut}_k(E, 0_E)$ is given by

$$\tau(a : b : c) = (b : a : c).$$

In this case, we have that

$$p - \tau(p) = 2p.$$

By Theorem 5,

$$A = \mathcal{A}(E, \sigma_p \tau) \text{ is AS-regular } \iff p \in E[6] \setminus E[3]$$

where $E[6] := \{q \in E \mid 6q = 0_E\}$. Since $|E[6]| = 36$ and $|E[3]| = 9$, $|E[6] \setminus E[3]| = 27$.

Example 2

The case of $j(E) = 0$

A generator $\tau \in \mathrm{Aut}_k(E, 0_E)$ is given by

$$\tau(a : b : c) = (b : a : \varepsilon c)$$

where ε is a primitive 3rd root of unity. In this case, we have that

$$\{p \in E \mid p - \tau(p) \in E[3]\} = E[3].$$

By Theorem 5,

$A = \mathcal{A}(E, \sigma_p \tau)$ is never AS-regular.

Example 3

The case of $j(E) = 12^3$

A generator $\tau \in \mathrm{Aut}_k(E, 0_E)$ is given by

$$\tau(a : b : c) = (\varepsilon^2 a + \varepsilon b + c : \varepsilon a + \varepsilon^2 b + c : a + b + c)$$

where ε is a primitive 3rd root of unity. In this case, we have that

$$\{p \in E \mid p - \tau(p) \in E[3]\} = E[3] \cup \{(1 : 1 : 1 + \sqrt{3}) + r \mid r \in E[3]\}.$$

By Theorem 5,

$$A = \mathcal{A}(E, \sigma_p \tau) \text{ is AS-regular} \Longleftrightarrow p \in \{(1 : 1 : 1 + \sqrt{3}) + r \mid r \in E[3]\},$$

and $|\{(1 : 1 : 1 + \sqrt{3}) + r \mid r \in E[3]\}| = 9.$

Classify up to graded algebra isomorphism

Theorem 6 [Itaba-M, 2018]

Let $A = \mathcal{A}(E, \sigma_p \tau^i)$ and $B = \mathcal{A}(E, \sigma_q \tau^j)$ be two geometric algebras where $p, q \in E \setminus E[3]$ and $i, j \in \mathbb{Z}_{|\tau|}$. Then $A \cong B$ if and only if

1. $i = j$, and,
2. there exist $r \in E[3]$ and $l \in \mathbb{Z}_{|\tau|}$ such that $q = \tau^l(p) + r - \tau^i(r)$.

- (i) If $j(E) \neq 0, 12^3$, then there exist three algebras
- (ii) If $j(E) = 0$, then there exist three algebras
- (iii) If $j(E) = 12^3$, then there exist four algebras

other than Sklyanin algebras up to graded algebra isomorphism.