Noncommutative Matrix Factorizations and Knörrer's Periodicity Theorem

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Motivations

k : an algebraically closed field of characteristic not 2.

The goal of our project :
to study and classify $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$ for "nice" noncommutative graded algebras A.

In the commutative case:

- $S = k[x_1, \ldots, x_n] \Rightarrow \operatorname{Proj} S = \mathbb{P}^{n-1} \Rightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(S) = \{0\}$.
- $A = S/(f), f \in S_d \Rightarrow \operatorname{Proj} A \subset \mathbb{P}^{n-1}$: a hypersurface of degree d $\Rightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(S) =???\ d = 2???$

- Knörrer's periodicity theorem is a powerful tool to calculate $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$.

Example

If $\mathrm{Proj}\, A \subset \mathbb{P}^{n-1}$ is a smooth quadric hypersurface, then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \begin{cases} \mathcal{D}^b(\mathrm{mod}\, k) & \text{if } n \text{ is odd,} \\ \mathcal{D}^b(\mathrm{mod}\, k^2) & \text{if } n \text{ is even.} \end{cases}$$

- Matrix factorizations are essential to prove Knörrer's periodicity theorem.

The aim of the talk:

1. Define a noncommutative matrix factorization.
2. Show a noncommutative version of Knörrer's periodicity theorem.
3. Classify $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$ for noncommutative smooth quadric hypersurfaces.

Matrix Factorizations

Definition 1

Let S be a commutative ring and $f \in S$. A matrix factorization of f is an ordered pair of matrices $(\Phi, \Psi) \in M_r(S) \times M_r(S)$ for some $r \in \mathbb{N}$ such that $\Phi\Psi = \Psi\Phi = fE_r$.

$\mathrm{MF}_S(f)$: the category of matrix factorizations of f.

Remark

1. In general, $(\Phi, \Psi) \neq (\Psi, \Phi)$.
2. $(1, f), (f, 1) \in S \times S$ are called trivial factorizations of f.

Example

$f = x^2 - yz \in k[x,y,z] \Rightarrow \operatorname{Proj} k[x,y,z]/(f) \subset \mathbb{P}^2$: a smooth quadric curve $\Rightarrow$ Essentially one non-trivial matrix factorization:

$$\begin{pmatrix} x & y \\ -z & -x \end{pmatrix}^2 = fE_2.$$

$\Rightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\operatorname{mod} k)$.

Example

$f = xw - yz \in k[x,y,z,w] \Rightarrow \operatorname{Proj} k[x,y,z,w]/(f) \subset \mathbb{P}^2$: a smooth quadric surface $\Rightarrow$ Essentially two non-trivial matrix factorizations:

$$\begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} w & -y \\ -z & x \end{pmatrix} = \begin{pmatrix} w & -y \\ -z & x \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = fE_2.$$

$\Rightarrow \underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\operatorname{mod} k^2)$.

Definition 2

Let S be a ring and $f \in S$. A noncommutative matrix factorization of f is a sequence of matrices $\{\Phi^i\}_{i\in\mathbb{Z}}$ in $M_r(S)$ for some $r \in \mathbb{N}$ such that $\Phi^i\Phi^{i+1} = fE_r$ for every $i \in \mathbb{Z}$.

$\mathrm{NMF}_S(f)$: the category of noncommutative matrix factorizations of f.

Remark

1. $(1, f) = \{\Phi^i\} \in \mathrm{NMF}_S(f)$ defined by $\Phi^{2i} = 1, \Phi^{2i+1} = f$, and $(f, 1) = \{\Psi^i\} \in \mathrm{NMF}_S(f)$ defined by $\Psi^{2i} = f, \Psi^{2i+1} = 1$ are called trivial factorizations of f.
2. If S is commutative and $(\Phi, \Psi) \in \mathrm{MF}_S(f)$, then $\{\Phi^i\} \in \mathrm{NMF}_S(f)$ where $\Phi^{2i} = \Phi, \Phi^{2i+1} = \Psi$. In fact, $\mathrm{MF}_S(f) = \mathrm{NMF}_S(f)$.

Noncommutative Quadric Hypersurfaces

Definition 3

Let S be a ring.

1. $f \in S$ is regular if, for $a \in S$, $af = 0$ or $fa = 0$ implies $a = 0$.
2. $f \in S$ is normal if $Sf = fS$.

Remark

1. S is a domain if and only if every non-zero element is regular.
2. A central element is normal, so if S is commutative, then every element is normal.

Definition 4 (Artin-Schelter, Smith-Van den Bergh)

Let S be a noetherian connected graded algebra (i.e. $S_0 = k$).

1. S is an n-dimensional AS-regular algebra if
   - $\operatorname{gldim} S = n < \infty$, and
   - $\operatorname{Ext}^i_S(k, S) \cong \begin{cases} 0 & \text{if } i \neq n, \\ k & \text{if } i = n. \end{cases}$
2. S is an n-dimensional quantum polynomial algebra if
   - S is an n-dimensional AS-regular algebra, and
   - $H_S(t) = H_{k[x_1,\dots,x_n]}(t) = (1-t)^{-n}$.
3. If S is an n-dimensional quantum polynomial algebra, then $\operatorname{Proj} S$ is called a quantum $\mathbb{P}^{n-1}$.
4. Moreover, if $f \in S_d$ is a regular normal element, and $A = S/(f)$, then $\operatorname{Proj} A \subset \operatorname{Proj} S$ is called a noncommutative hypersurface of degree d.
5. In particular, if $d = 2$, then $\operatorname{Proj} A \subset \operatorname{Proj} S$ is called a noncommutative quadric hypersurface.

Eisenbud's Theorem

Definition 5

Let A be a ring.

1. An A-module M is maximal Cohen-Macaulay if $\mathrm{Ext}^i_A(M, A) = 0$ for every $i \geq 1$.
2. An A-module M is totally reflexive if
   - $M \in \mathrm{CM}(A)$.
   - $\mathrm{Hom}_A(M, A) \in \mathrm{CM}(A^{op})$.
   - $M \cong \mathrm{Hom}_{A^{op}}(\mathrm{Hom}_A(M, A), A)$.

$\mathrm{CM}(A)$: the category of finitely generated maximal Cohen-Macaulay modules.

$\mathrm{TR}(A)$: the category of finitely generated totally reflexive modules.

Note that $\mathrm{add}\{A\} \subset \mathrm{TR}(A) \subset \mathrm{CM}(A)$.

Theorem 6 (Eisenbud)

If S is a commutative noetherian regular local ring, $f \in S$ and $A = S/(f)$, then

$$\mathrm{MF}_S(f)/\operatorname{add}\{(1,f)\} \cong \mathrm{CM}(A)$$
$$\underline{\mathrm{MF}}_S(f) := \mathrm{MF}_S(f)/\operatorname{add}\{(1,f),(f,1)\} \cong \mathrm{CM}(A)/\operatorname{add}\{A\} =: \underline{\mathrm{CM}}(A).$$

Theorem 7 (MU)

If S is a noetherian ring, $f \in S$ is a regular normal element, and $A = S/(f)$, then there are fully faithful embeddings

$$\mathrm{NMF}_S(f)/\operatorname{add}\{(1,f)\} \to \mathrm{TR}(A),$$
$$\underline{\mathrm{NMF}}_S(f) := \mathrm{NMF}_S(f)/\operatorname{add}\{(1,f),(f,1)\} \to \mathrm{TR}(A)/\operatorname{add}\{A\} =: \underline{\mathrm{TR}}(A).$$

Theorem 8 (MU, Cassidy-Conner-Kirkman-Moore)

If S is a graded quotient algebra of an AS-regular algebra, $f \in S_d$ is a regular normal element, and $A = S/(f)$, then

$$\mathrm{NMF}_S^{\mathbb{Z}}(f)/\operatorname{add}^{\mathbb{Z}}\{(1,f)\} \cong \mathrm{TR}_S^{\mathbb{Z}}(A)$$
$$\underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) \cong \underline{\mathrm{TR}}_S^{\mathbb{Z}}(A)$$

where $\mathrm{TR}_S^{\mathbb{Z}}(A) := \{M \in \mathrm{TR}^{\mathbb{Z}}(A) \mid \mathrm{pd}_S(M) < \infty\}$.
In particular, if S is an AS-regular algebra, then

$$\mathrm{NMF}_S^{\mathbb{Z}}(f)/\operatorname{add}^{\mathbb{Z}}\{(1,f)\} \cong \mathrm{CM}^{\mathbb{Z}}(A),$$
$$\underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(A).$$

Remark

If S is a commutative noetherian connected graded algebra, then $S \cong k[x_1, \dots, x_n]/I$ is a graded quotient algebra of an AS-regular algebra $k[x_1, \dots, x_n]$, so the above theorem always applies.

Knörrer's Periodicity Theorem

Theorem 9 (Knörrer)

If $S = k[[x_1, \ldots, x_n]]$ *and* $f \in (x_1, \ldots, x_n)^2$, *then*

$$\underline{\mathrm{CM}}(S/(f)) \cong \underline{\mathrm{MF}}_S(f) \cong \underline{\mathrm{MF}}_{S[u,v]}(f+u^2+v^2) \cong \underline{\mathrm{CM}}(S[u,v]/(f+u^2+v^2)).$$

Definition 10

Let S be a ring and σ a ring automorphism of S. The Ore extension of S by σ is a ring $S[u;\sigma] = S[u]$ as a free right S-module such that $au = u\sigma(a)$ for every $a \in S$.

Theorem 11 (MU)

Let S *be a noetherian ring and* $f \in S$ *a regular normal element. If* σ, τ *are ring automorphisms of* S *such that* $\sigma(f) = \tau(f) = f$ *and* $af = f\sigma(\tau(a)) = f\tau(\sigma(a))$ *for every* $a \in S$, *then there is a fully faithful embedding* $\underline{\mathrm{NMF}}_S(f) \to \underline{\mathrm{NMF}}_{S[u;\sigma][v;\tau]}(f + uv)$.

Theorem 12 (MU, He-Ye)

Let S *be an AS-regular algebra and* $f \in S_{2e}$ *a regular normal element. If there exists a graded algebra automorphism* σ *of* S *such that* $\sigma(f) = f$ *and* $af = f\sigma^2(a)$ *for every* $a \in S$*, then*

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(S/(f)) \cong \underline{\mathrm{NMF}}_S^{\mathbb{Z}}(f) \cong$$
$$\underline{\mathrm{NMF}}_{S[u;\sigma][v;\sigma]}^{\mathbb{Z}}(f + u^2 + v^2) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}(S[u;\sigma][v;\sigma]/(f + u^2 + v^2))$$

where $\deg u = \deg v = e$.

Remark

1. The above technical conditions are needed to guarantee $f + u^2 + v^2 \in S[u;\sigma][v;\sigma]$ is a homogeneous normal element.
2. If $f \in S_{2e}$ is central, then we may take $\sigma = \mathrm{id}_S$, so if $S = k[x_1, \ldots, x_n]$, then the above theorem always applies.

Example

Let $A = k[x_1, \dots, x_n]/(f)$ where $f \in k[x_1, \dots, x_n]_2$. If $\operatorname{Proj} A \subset \mathbb{P}^{n-1}$ is smooth, then $A \cong k[x_1, \dots, x_n]/(x_1^2 + \cdots + x_n^2)$, so

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \begin{cases} \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1]/(x_1^2)) \cong \mathcal{D}^b(\operatorname{mod} k) & \text{if } n \text{ is odd,} \\ \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x_1, x_2]/(x_1^2 + x_2^2)) \cong \mathcal{D}^b(\operatorname{mod} k^2) & \text{if } n \text{ is even.} \end{cases}$$

Theorem 13 (MU)

If S is a quantum polynomial algebra, and $f \in S_2$ is a regular central element, then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}\left(\frac{S[u;-1][v;-1]}{(f+u^2+v^2)}\right) \cong \underline{\mathrm{CM}}^{\mathbb{Z}}\left(\frac{S[u]}{(f+u^2)}\right) \times \underline{\mathrm{CM}}^{\mathbb{Z}}\left(\frac{S[v]}{(f+v^2)}\right).$$

Example

If $S = k\langle x,y,z\rangle/(yz+zy, zx+xz, xy+yx) = k[x][y;-1][z;-1]$, and $A = S/(x^2+y^2+z^2)$, then $\mathrm{Proj}\, A$ is "smooth", but

$$\begin{aligned}\underline{\mathrm{CM}}^{\mathbb{Z}}(A) &\cong \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x,y]/(x^2+y^2)) \times \underline{\mathrm{CM}}^{\mathbb{Z}}(k[x,z]/(x^2+z^2)) \\ &\cong \mathcal{D}^b(\mathrm{mod}\, k^2) \times \mathcal{D}^b(\mathrm{mod}\, k^2) \cong \mathcal{D}^b(\mathrm{mod}\, k^4).\end{aligned}$$

Rank

Definition 14

Let S be a graded algebra. For $f \in S_2$, we define the rank of f over S by

$$\operatorname{rank}_S f := \min\{r \in \mathbb{N}^+ \mid f = u_1v_1 + \cdots + u_rv_r, 0 \neq u_i, v_i \in S_1\}.$$

Let S be an n-dimensional quantum polynomial algebra. We say that f is of the highest rank if

$$\operatorname{rank}_S f = \operatorname{rank}_{k[x_1,\dots,x_n]}(x_1^2 + \cdots + x_n^2) = \begin{cases} (n+1)/2 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

Remark

$\operatorname{rank}_S f \geq 2$ if and only if f is an irreducible element in S.

Theorem 15 (MU)

Let S be an n-dimensional quantum polynomial algebra, $f \in S_2$ a regular normal element of the highest rank, and $A = S/(f)$. If $\operatorname{Proj} A$ is "smooth", then

$$\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \begin{cases} \mathcal{D}^b(\operatorname{mod} k) & \text{if } n = 1, 3, 5, \\ \mathcal{D}^b(\operatorname{mod} k^2) & \text{if } n = 2, 4, 6. \end{cases}$$

Example 16

If $S = k\langle x, y, z\rangle/(yz + zy, zx + xz, xy + yx)$, $f = x^2 + y^2 + z^2 \in S_2$, and $A = S/(f)$, then $\operatorname{Proj} A$ is "smooth", but $\underline{\mathrm{CM}}^{\mathbb{Z}}(A) \cong \mathcal{D}^b(\operatorname{mod} k^4)$.
Since $f = (x + y + z)^2$, $\operatorname{rank}_S f = 1$, so f is not of the highest rank.
Isomorphism classes of all non-trivial indecomposable matrix factorizations of f are given by

$$(x + y + z)^2 = (x + y - z)^2 = (x - y + z)^2 = (x - y - z)^2 = f.$$

We are able to give a complete classification of $\underline{\mathrm{CM}}^{\mathbb{Z}}(A)$ by using graphical methods in the following case:

1. $S = k\langle x_1, \ldots, x_n\rangle/(x_i x_j - \varepsilon_{ij} x_j x_i)_{1 \le i < j \le n}$ is a skew polynomial algebra,
2. $f = x_1^2 + \cdots + x_n^2 \in S_2$ is a normal element,
3. $A = S/(f)$, and
4. $n \le 6$.

(The next talk by Ueyama!!)