On Thompson's group F and its group algebra

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In this talk,

- we first introduce an application of (undirected) two edge-colored graphs to group algebras of groups which have non-abelian free subgroups.

We have used these graphs to study primitivity of group algebras of non-Noetherian groups, where generally a ring R is right primitive if it has a faithful irreducible right R-module .

Our method using two edge-colored graphs seems to be effective to investigate a group algebra if its group has non-abelian free subgroups.

But there exist some non-Noetherian groups with no non-abelian free subgroups; for example Thompson's group F and a free Burnside group of large exponent.

- Next we introduce briefly Thompson's group F and consider amenability of it.

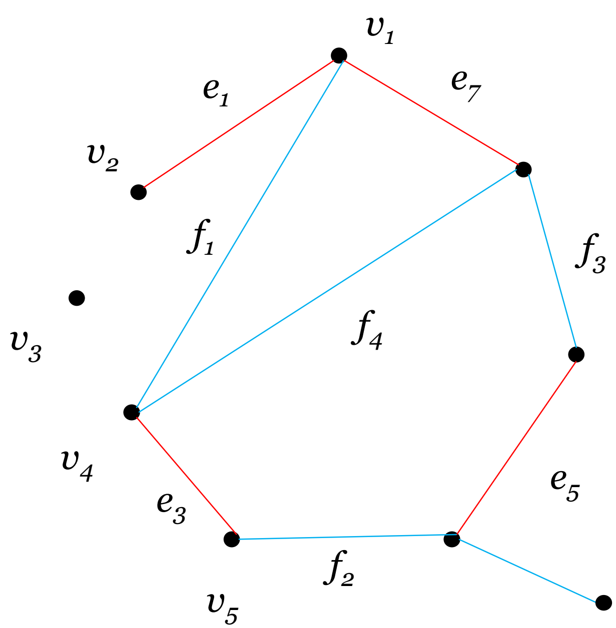
- Finally, in order to be able to investigate the group algebra of this group, we use a 'directed' two edge-colored graph and improve our method.

Two-edge coloured graphs

A two-edge colored graph is a simple graph each of whose edges colored with one of two different colors.

$V = \{ v_1, v_2, \ldots, v_n \}$ is a vertex set, E and F are two edge sets;

$$E = \{e_1, e_2, \ldots, e_m \} \qquad F = \{f_1, f_2, \ldots, f_m \}$$

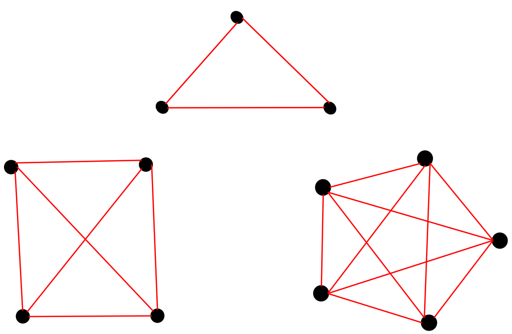


A cycle in the graph is called an alternating cycle if its edges belong alternatively to E and F. For example, $f_1 e_3 f_2 e_5 f_3 e_7$

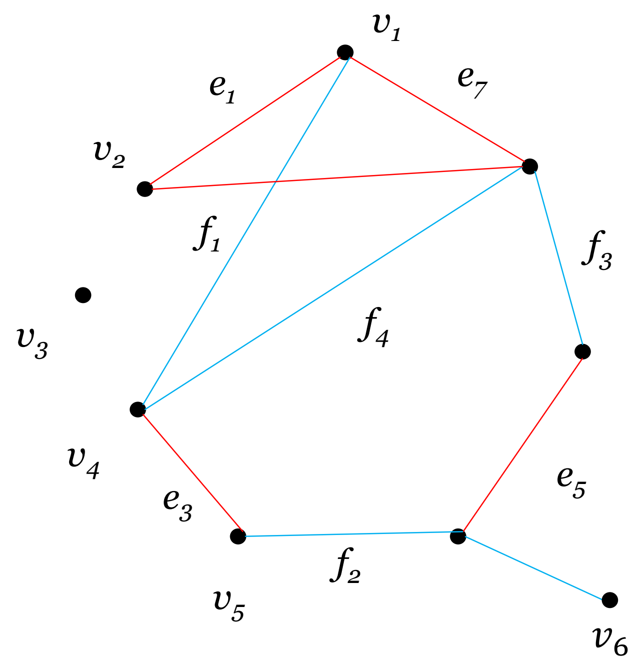
SR-graphs

A two-edge colored graph $S = (V, E, F)$ is an SR-graph if every component of $\mathcal{G} = (V, E)$ is a complete graph.

In an SR-graph, we call an alternating cycle an SR-cycle; for example, $f_1 e_3 f_2 e_5 f_3 e_7$.



Complete graphs



$I(\mathcal{G}) = \{v_3, v_6\}$

$E = \{e_1, e_2, \ldots, e_m\}$

$F = \{f_1, f_2, \ldots, f_m\}$

An introduction to an application of SR-graph theory

We begin with the following simple problem.

Let G be a group and KG the group algebra of G over a field K.

We denote $KG \setminus \{0\}$, the non-zero elements in KG, by KG^*.

Problem 1

Find elements $A, B \in KG^*$ such that

$$AX + BY \neq 0 \text{ for any } X, Y \in KG^*.$$

<u>If G has a non-abelian free subgroup, then we can find this kind of elements.</u>

Let G be a group which has a nonabelian free subgroup.

In this case, G has always a free subgroup of infinite rank: $\langle a_1, a_2, b_1, b_2, \cdots \rangle$.

Let $A = a_1 + a_2$ and $B = b_1 + b_2$. Suppose, to the contrary, that $AX + BY = 0$ for some $X, Y \in KG^*$.

Since $X, Y \in KG^*$, they are expressed as follows:

$$X = \sum_{x \in S_X} \alpha_x x,\ Y = \sum_{y \in S_Y} \beta_y y,$$

where $\alpha_x, \beta_y \in K \setminus \{0\}$, $S_X = Supp(X)$ and $S_Y = Supp(Y)$.

Since $AX + BY = 0$, we have

$$\sum_{x \in S_X} \alpha_x (a_1 x + a_2 x) + \sum_{y \in S_Y} \beta_y (b_1 y + b_2 y) = 0.$$

We would like to regard these elements $a_i x$ and $b_i y$ as vertices. Because of that, we need to distinguish all these elements even if for $i \neq j$, $a_i x = a_j x'$, $b_i y = b_j y'$ or $a_i x = b_j y$ in G.

So we define the vertex set and two edge set as follows:

$$V = \{(a_i, x), (b_i, y) \mid i = 1, 2, x \in S_X, y \in S_Y\}$$

$$E = \{vw \mid v, w \in V; v \neq w, [v] = [w] \text{ in } G\}, \text{ where } [v] = ax \text{ if } v = (a, x).$$

$$F = \{vw \mid v, w \in V; v \neq w, v = (a_1, x), w = (a_2, x) \text{ or } v = (b_1, y), w = (b_2, y)\}$$

$$V = \{(a_i, x), (b_i, y) \mid i = 1, 2, x \in S_X, y \in S_Y\}$$

$$E = \{vw \mid v, w \in V; v \neq w, [v] = [w] \text{ in } G\}, \text{ where } [v] = ax \text{ if } v = (a, x).$$

$$F = \{vw \mid v, w \in V; v \neq w, v = (a_1, x), w = (a_2, x) \text{ or } v = (b_1, y), w = (b_2, y)\}$$

Since $\displaystyle\sum_{x \in S_X} \alpha_x (a_1 x + a_2 x) + \sum_{y \in S_Y} \beta_y (b_1 y + b_2 y) = 0$,

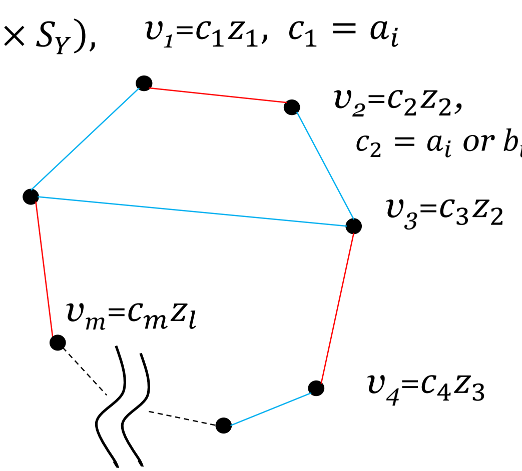
all elements of G in this equation are cancelled each other.

Hence $(c_1, z_1) \in \{a_1, a_2\} \times S_X, \exists (c_2, z_2) \in (\{a_1, a_2\} \times S_X) \cup (\{b_1, b_2\} \times S_Y)$,

$$c_1 z_1 = c_2 z_2,$$

If $c_2 = a_i$(resp. $c_2 = b_i$), then for $i \neq j$, $a_j z_2$ (resp. $b_j z_2$) exists in the above expression, and so

$\exists c_3 \in \{a_1, a_2, b_1, b_2\}$ with $c_3 \neq c_2$ and $\exists (c_4, z_3) \in (\{a_1, a_2\} \times S_X) \cup (\{b_1, b_2\} \times S_Y)$,



$$c_3 z_2 = c_4 z_3,$$

$$\vdots$$

$$c_m z_l = c_{m+1} z_1,$$

$$\Longrightarrow \quad c_1^{-1} c_2 c_3^{-1} c_4 \cdots c_m^{-1} c_{m+1} = 1,$$

where $c_i \in \{a_1, a_2, b_1, b_2\}$ and $c_i \neq c_{i+1}$.

Since $\{a_1, a_2, b_1, b_2\}$ is a free basis, this implies a contradiction. ■

We have thus seen that $A = a_1 + a_2, B = b_1 + b_2 \Longrightarrow AX + BY \neq 0$ for any $X, Y \in KG^*$.

Problem 1 is strongly connected with amenability of groups.

$\exists A, B \in KG^*, \forall X, Y \in KG^*, AX + BY \neq 0 \Longrightarrow G$ is not amenable.

▸ G has a non-abelian free subgroup $\Longrightarrow G$ is not amenable.

▸(Definition) G is amenable if for $P(G) = \{S | S \subseteq G\}, \exists \mu: P(G) \to [0,1]$ such that
1. $\mu(G) = 1$.
2. If S and T are disjoint subsets of G, then $\mu(S \cup T) = \mu(S) + \mu(T)$
3. If $S \in P(G)$ and $g \in G$, then $\mu(gS) = \mu(S)$.

▸Finite groups and abelian groups are amenable.

▸The Burnside group $B(m, n)$ is not amenable if $m > 1$ and n is enough large.

▸ F is amenable$\Longrightarrow \forall A, B \in KG^*, \exists X, Y \in KG^*, AX = BY$ for any $X, Y \in KG^*$.

▸Is Thompson's group F is amenable?

Thompson's group *F*

$$F = \langle x_0, x_1, \cdots, x_i, \cdots \mid x_i^{-1} x_j x_i = x_{j+1},\ \text{for } i < j \rangle.$$

$$= \langle x_0, x_1 \mid [x_0\, x_1^{-1},\ x_0^{-1} x_1 x_0], [x_0\, x_1^{-1},\ x_0^{-2} x_1 x_0^2] \rangle.$$

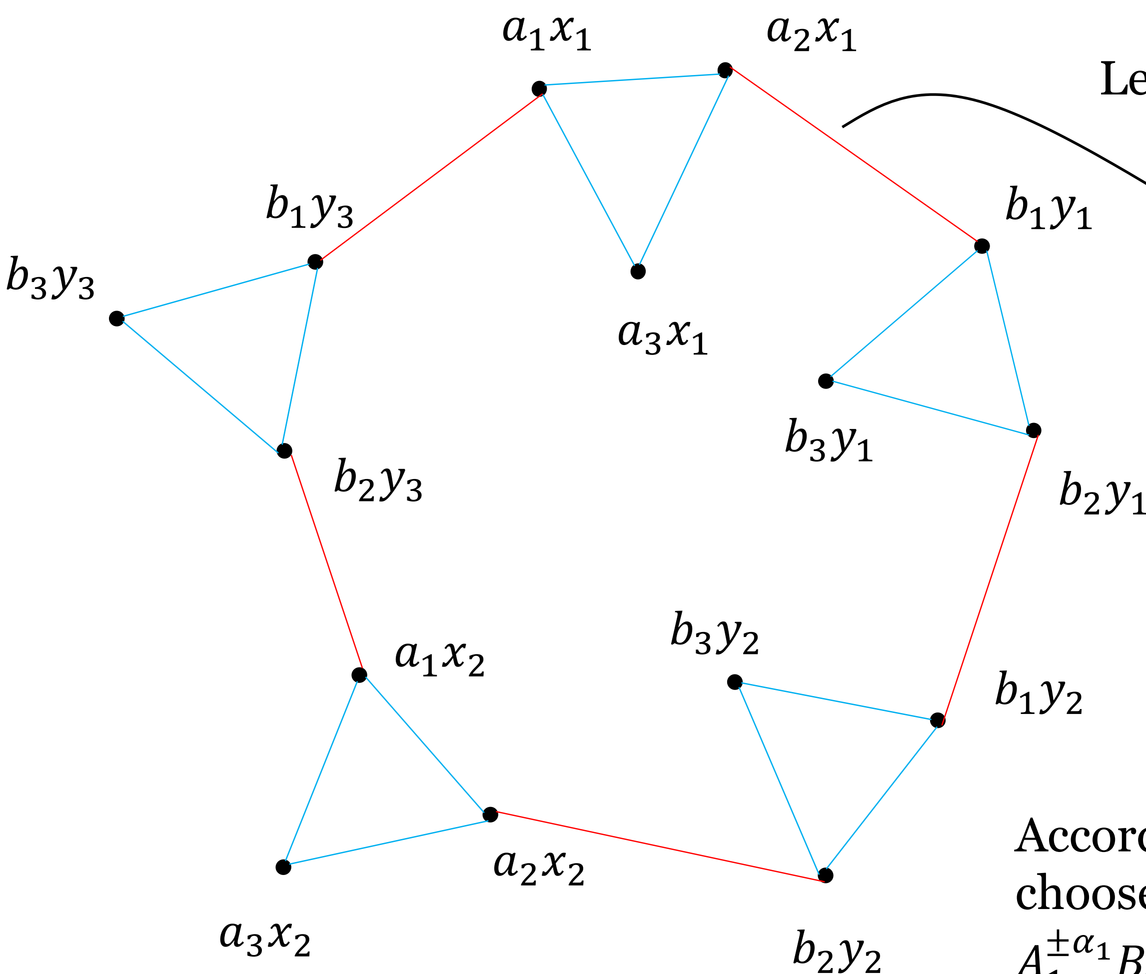
- F is non-noetherian
- <u>F has no non-abelian free subgroups.</u>
- F is a torsion free group and includes a free subsemigroup.
- $\exists T \supset F$ such that T is simple.

We need to improve our graph theory so as to be effective for Thompson group *F;* generally for a non-Noetherian group with no free subgroup.

5. Improvement on SR-graph theory

We considered the following SR-graph. We set here

$$A = a_1 + a_2 + a_3 \text{ and } B = b_1 + b_2 + b_3.$$



Let $A_1 = a_1a_2^{-1}$ and $B_1 = b_1b_2^{-1}$.

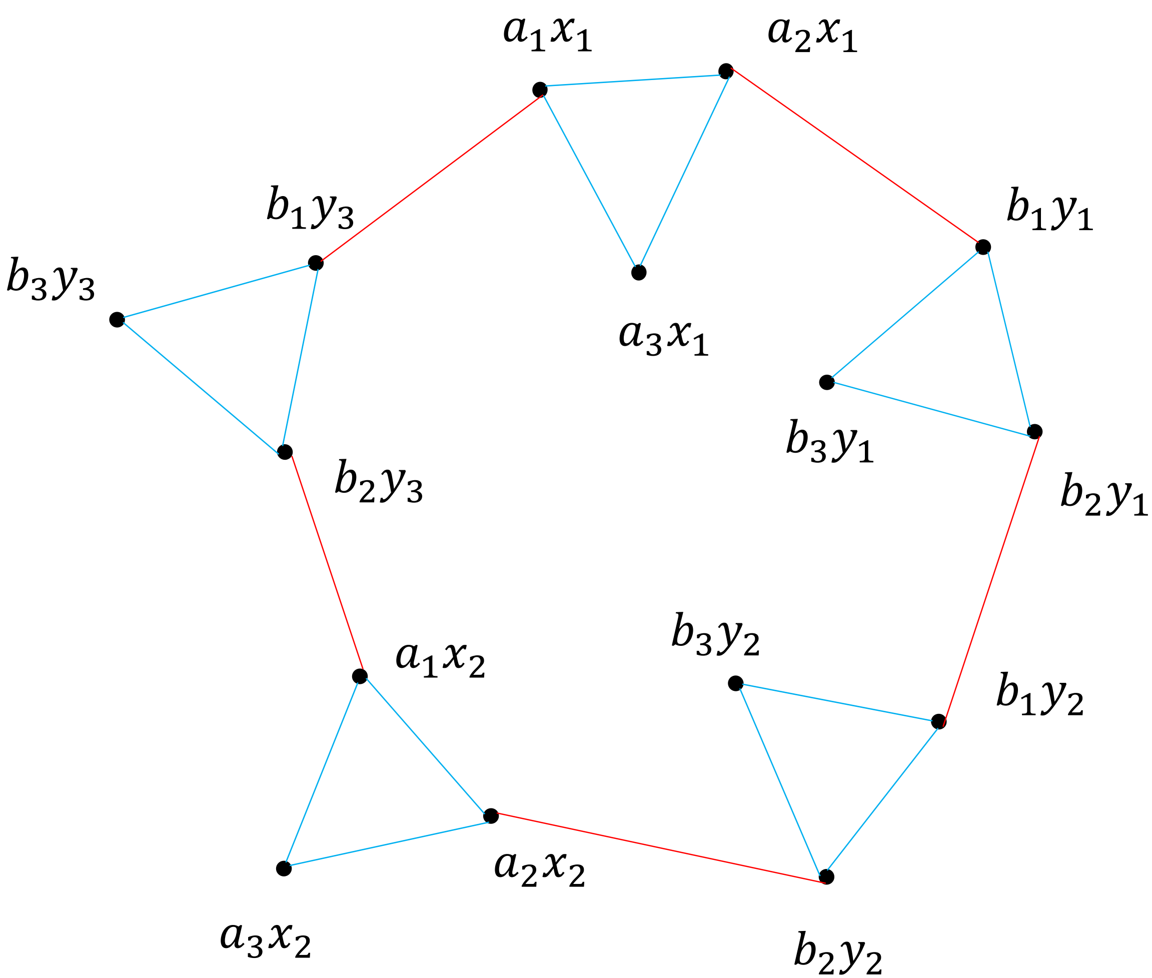
$$A_1B_1B_1A_1^{-1}B_1^{-1} = 1$$

We have to choose A_1 and B_1 so as to $A_1B_1B_1A_1^{-1}B_1^{-1} \neq 1$

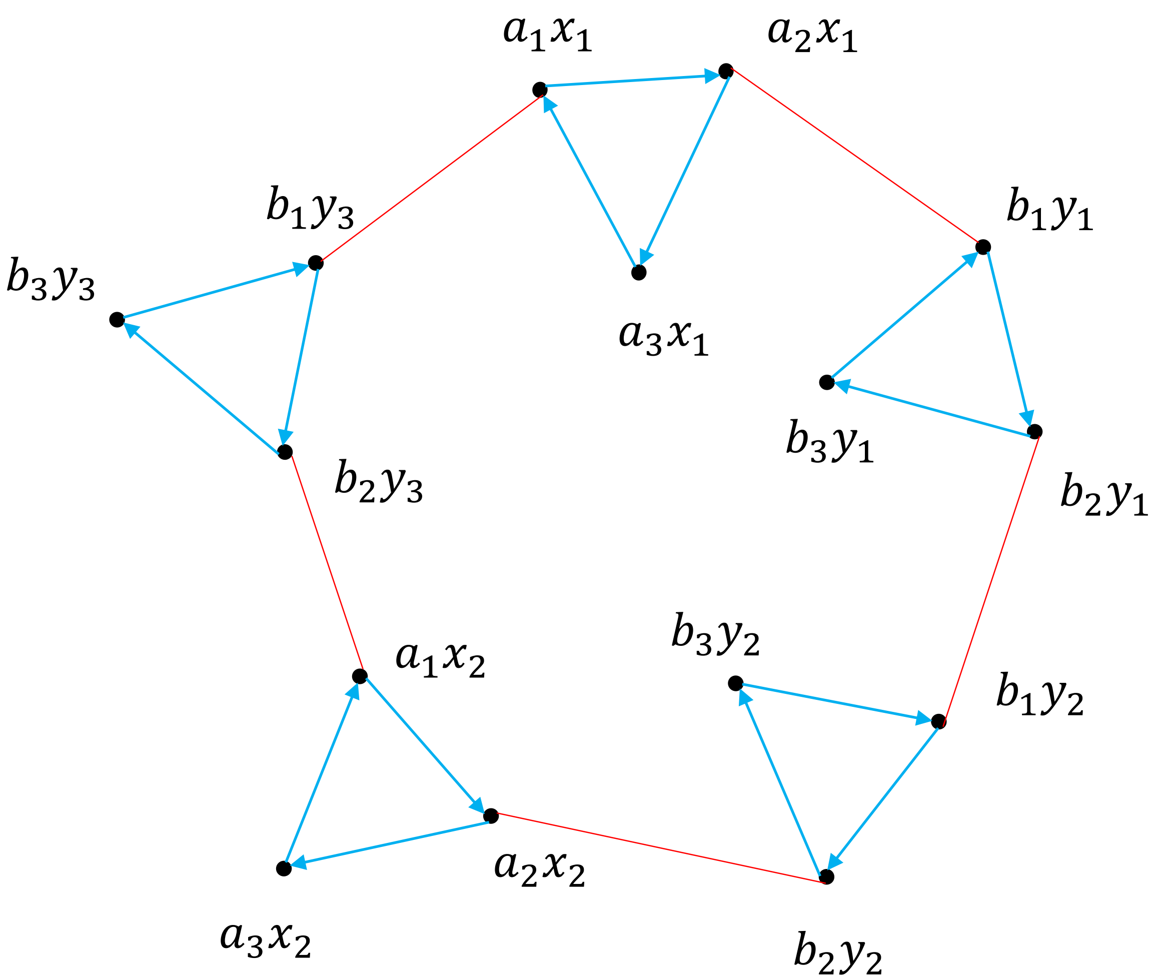
Generally, it may happen that

$$A_1^{\pm\alpha_1}B_1^{\pm\beta_1} \cdots A_1^{\pm\alpha_m}B_1^{\pm\beta_m} = 1$$

According to our method, we have to choose A_1 and B_1 so as to $A_1^{\pm\alpha_1}B_1^{\pm\beta_1} \cdots A_1^{\pm\alpha_m}B_1^{\pm\beta_m} \neq 1$

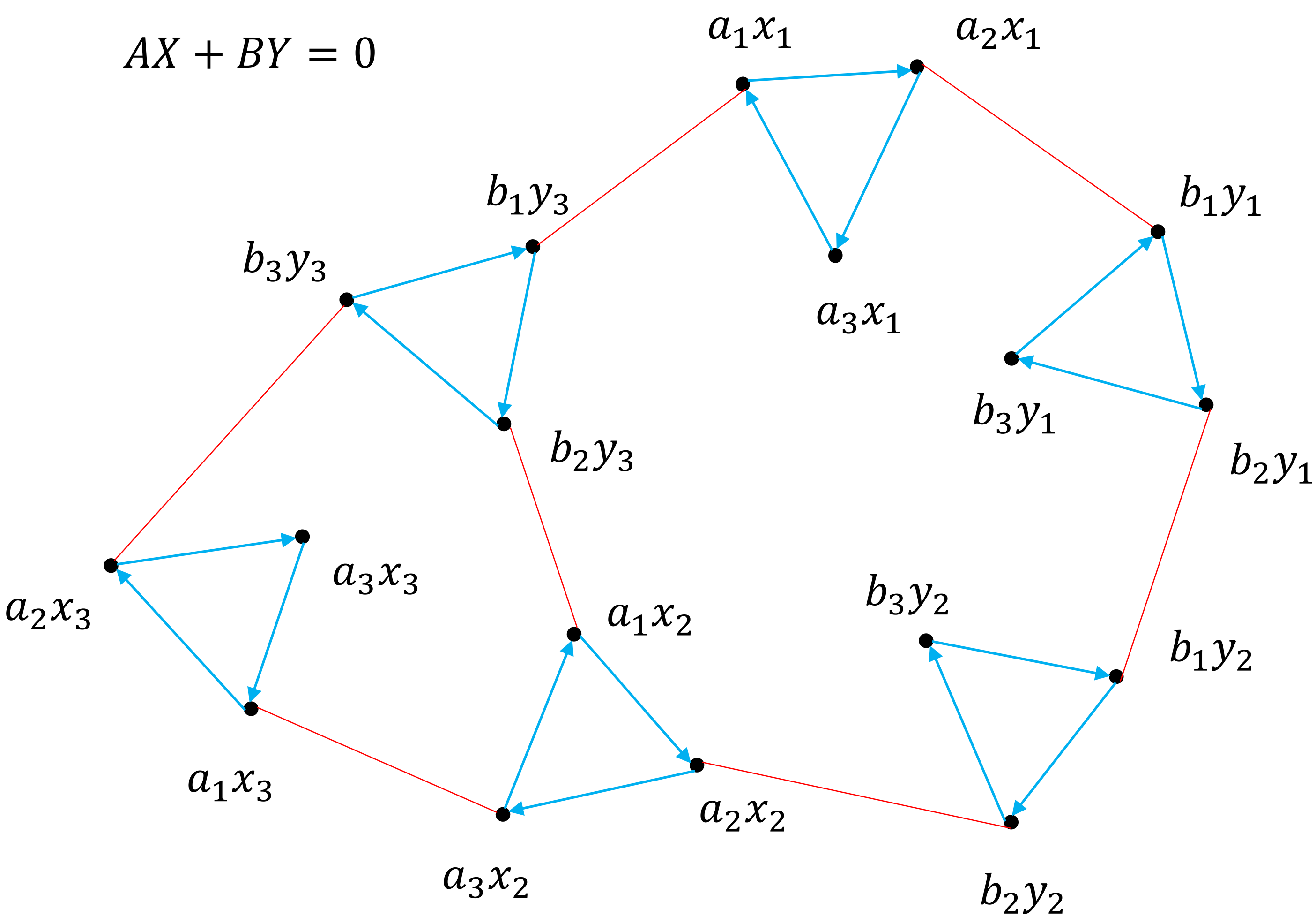


We replace an undirected SR-graph with a directed one.



In this graph, an SR-cycle means a cycle along the direction of arrows.

$AX + BY = 0$



Let $A_1 = a_1a_2^{-1}$, $A_2 = a_2a_3^{-1}$, $B_1 = b_1b_2^{-1}$ and $B_3 = B_3B_1^{-1}$.

$A_1B_1B_1A_2A_1B_3 = 1$; We have only to choose $A_1B_1B_1A_2A_1B_3 \neq 1$.

A result: If F satisfies

$$\left[\begin{array}{l} \exists a_i, b_i \in F \setminus \{1\} \ (i = 1,2,3) \text{ such that } \ u_1 u_2 \ \cdots u_n = 1 \\ \qquad \text{for } \ u_i \in \{a_1 a_2^{-1}, a_2 a_3^{-1}, a_3 a_1^{-1}, b_1 b_2^{-1}, b_2 b_3^{-1}, b_3 b_1^{-1}\}. \\ \qquad \Longrightarrow \begin{array}{l} \exists i, \ u_i = c_j c_k^{-1}, \ \ u_{i+1} = c_k c_l^{-1}, \\ \text{where } c_i \in \{a_1, a_2, a_3, b_1, b_2, b_3\}, \end{array} \end{array}\right.$$

then two elements $A = a_1 + a_2 + a_3$ and $B = b_1 + b_2 + b_3$ satisfy

$\forall X, Y \in KG^*, \ AX + BY \neq 0.$

In particular, F is not amenable.

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Thank you!