The Gabriel-Quillen functor for extriangulated categories

Yasuaki Ogawa

Nagoya University

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Notation

Categories and functors are always assumed to additive.

- $\mathcal{C}$ - category.
- $\mathsf{Mod}\,\mathcal{C}$ - category of contravariant functors: $\mathcal{C} \to \mathsf{Ab}$.

$$\begin{array}{ccc} \mathsf{Mod}\,\mathcal{C} & \supseteq & \mathsf{Lex}\,\mathcal{C} \\ \cup\mathrel{|} & & \cup\mathrel{|} \\ \mathsf{mod}\,\mathcal{C} & \supseteq & \mathsf{lex}\,\mathcal{C} \end{array}$$

- The Yoneda functor is denoted by $\mathbb{Y} : \mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C}$.

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$$\begin{array}{ccc}
\mathsf{Mod}\,\mathcal{C} & \supseteq & \underset{\text{left exact}}{\mathsf{Lex}\,\mathcal{C}} \\
\cup\!| & & \cup\!| \\
\underset{\text{finitely presented}}{\mathsf{mod}\,\mathcal{C}} & \supseteq & \mathsf{lex}\,\mathcal{C}
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Contents

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Let $(\mathcal{C}, \mathbb{E})$ be a small exact category.

1. There exists the following localization sequence

$$\begin{tikzcd} \mathsf{Mod}\,\mathcal{C} \arrow[r, "Q"] & \mathsf{Lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

   where R is the canonical inclusion.

2. The composed functor $E_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{Mod}\,\mathcal{C} \xrightarrow{Q} \mathsf{Lex}\,\mathcal{C}$ is exact and fully faithful.
3. $\mathcal{C}$ is extension-closed in $\mathsf{Lex}\,\mathcal{C}$ and the exact structure $\mathbb{E}$ is a class of all short sequences which belong to $\mathcal{C}$.

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Closer look at the GQ theorem

Definition (Auslander)

Let $\delta : Z \to Y \to X$ be a conflation in $\mathcal{C}$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{C}(-,Z) \longrightarrow \mathcal{C}(-,Y) \longrightarrow \mathcal{C}(-,X)$$

in $\mathsf{Mod}\,\mathcal{C}$.

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Definition (Auslander'66)

Define the following subcategories in $\mathsf{Mod}\,\mathcal{C}$.

- $\mathsf{def}\,\mathcal{C}$ - all defects in $\mathsf{Mod}\,\mathcal{C}$.
- $\mathsf{Def}\,\mathcal{C}$ - all filtered colimits of factors of defects in $\mathsf{Mod}\,\mathcal{C}$.

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where R is the canonical inclusion.

The definition

The notion of extriangulated category is a simultaneous generalization of triangulated ones and exact ones.

Definition (Nakaoka-Palu'19)

The extriangulated category is a triple $(\mathcal{C}, \mathbb{E}, \mathfrak{s})$, where

- $\mathcal{C}$ - an additive category;
- $\mathbb{E}$ - a biadditive functor $\mathcal{C} \times \mathcal{C}^{\mathsf{op}} \to \mathsf{Ab}$;
- $\mathfrak{s}$ - an assignment from an element in $\mathbb{E}(X, Z)$ to a sequence $Z \to Y \to X$ in $\mathcal{C}$,

with some suitable compatibility.

Some examples

- Extension-closed subcategory in a triangulated category has a natural extriagulated structure.
- Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in a triangulated category. Then $\mathcal{U}$ has a natural extriangulated structure.

The GQ type localization sequence

Theorem (O)

Let $\mathcal{C}$ be a small extriangulated category. Then there exists a localization sequence

$$\begin{tikzcd} \mathsf{Def}\,\mathcal{C} \arrow[r, shift left] & \mathsf{Mod}\,\mathcal{C} \arrow[l, bend left] \arrow[r, "Q"] & \mathsf{Lex}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

where R is the canonical inclusion.

A finitely presented version

Lemma (Freyd'65)

The following are equivalent for an additive category $\mathcal{C}$:

1. The category $\mathcal{C}$ admits weak-kernels;
2. The full subcategory $\mathsf{mod}\,\mathcal{C}$ is an exact abelian subcategory in $\mathsf{Mod}\,\mathcal{C}$.

Proposition (O)

Let $\mathcal{C}$ be an extriangulated category with weak-kernels. Then $\mathsf{def}\,\mathcal{C}$ is a Serre subcategory in $\mathsf{mod}\,\mathcal{C}$.

Remark

The quotient functor $Q : \mathsf{mod}\,\mathcal{C} \to \frac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}}$ does not necessarily have a right adjoint.

The GQ functor

This situation is depicted as follows:

$$\operatorname{def}\mathcal{C} \longrightarrow \operatorname{mod}\mathcal{C} \xrightarrow{Q} \frac{\operatorname{mod}\mathcal{C}}{\operatorname{def}\mathcal{C}}$$

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$$\begin{tikzcd} \operatorname{def}\mathcal{C} \arrow[r] & \operatorname{mod}\mathcal{C} \arrow[l, dashed, bend left, "\color{red}\times"] \arrow[r, "Q"] & \dfrac{\operatorname{mod}\mathcal{C}}{\operatorname{def}\mathcal{C}} \arrow[l, dashed, bend left, "\color{red}\times"] \end{tikzcd}$$

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\mathsf{def}\,\mathcal{C} \arrow[r, shift left] & \mathsf{mod}\,\mathcal{C} \arrow[l, dashed, bend left, "\times" marking] \arrow[r, "Q", shift left] & \dfrac{\mathsf{mod}\,\mathcal{C}}{\mathsf{def}\,\mathcal{C}} \arrow[l, dashed, bend left, "\times" marking] \arrow[d, phantom, "\rotatebox{90}{$=:$}"] \\
& \mathcal{C} \arrow[u, "\mathbb{Y}"] \arrow[r, "Q \circ \mathbb{Y}"] & \mathsf{gq}\,\mathcal{C}
\end{tikzcd}
$$

We put $E_{\mathcal{C}} := Q \circ \mathbb{Y}$ and call it the *GQ functor* for $\mathcal{C}$.

A use of the GQ functor

Theorem (O)

The GQ functor $E_{\mathcal{C}} : \mathcal{C} \to \mathsf{gq}\,\mathcal{C}$ is:

1. exact and fully faithful iff $\mathcal{C}$ is an exact category;
2. an equivalence iff $\mathcal{C}$ is an abelian category.

Corollary (Auslander's defect formula)

Let $\mathcal{C}$ be an abelian category. Then we have the localization sequence

$$\begin{tikzcd} \mathsf{def}\,\mathcal{C} \arrow[r] & \mathsf{mod}\,\mathcal{C} \arrow[l, bend left] \arrow[r, "Q"] & \mathsf{gq}\,\mathcal{C} \arrow[l, bend left, "R"] \end{tikzcd}$$

In particular, we have an equality $\mathcal{C} = \mathsf{lex}\,\mathcal{C}$ in $\mathsf{mod}\,\mathcal{C}$.

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In particular, we have an equality $\mathcal{C} = \mathsf{lex}\,\mathcal{C}$ in $\mathsf{mod}\,\mathcal{C}$.

The exact case

Theorem

Let $\mathcal{C}$ be an idempotent complete exact category with weak-kernels. Assume that it has enough projctives.

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Torsion class

- $\mathcal{T}$ - triangulated category.
- $(\mathcal{U}, \mathcal{V})$ - torsion pair.

Proposition

There exists the following localization sequence.

$$\begin{tikzcd} \mathsf{def}\,\mathcal{U} \arrow[r] & \mathsf{mod}\,\mathcal{U} \arrow[r, "Q"] \arrow[l, bend left] & \mathsf{lex}\,\mathcal{U} \arrow[l, bend left, "R"] \end{tikzcd}$$

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The heart of $(\mathcal{U}, \mathcal{V})$

- $(\mathcal{U}, \mathcal{V})$ - torsion pair in $\mathcal{T}$.
- Put $\mathcal{W} := \mathcal{U} \cap \mathcal{V}[-1]$.
- Put $\mathcal{T}^+ := \mathcal{U}[-1] * \mathcal{W}, \mathcal{T}^- := \mathcal{W} * \mathcal{V}$ and $\mathcal{H} := \mathcal{T}^+ \cap \mathcal{T}^-$.
- $\underline{\mathcal{H}} := \mathcal{H}/[\mathcal{W}]$ - the Nakaoka heart.

Theorem (Nakaoka)

The Nakaoka heart $\underline{\mathcal{H}}$ is abelian.

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The heart of $(\mathcal{U}, \mathcal{V})$

Example

Let $(\mathcal{U}, \mathcal{V})$ be a torsion pair in $\mathcal{T}$.

- If $(\mathcal{U}, \mathcal{V}[-1])$ forms a t-structure, then $\underline{\mathcal{H}}$ is the usual heart of it.
- If $\mathcal{U}$ is a (2-)cluster tilting subcategory, then $\underline{\mathcal{H}}$ is $\mathcal{T}/[\mathcal{C}]$ which has been shown to be abelian by Koenig-Zhu.

Theorem (O)

The Nakaoka heart $\underline{\mathcal{H}}$ is equivalent to $\mathsf{lex}\,\mathcal{U}$.

Thank you for your attention!