Relationships between quantized algebras and their semiclassical limits

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Motivation

What is motion?

1. Poisson algebras appear in classical mechanical system

2. Quantized algebras appear in quantum mechanical system

Poisson algebra 1

Equations of motion of mechanical system are expressed by Hamiltonian systems to develop the theory of classical mechanical system:

y: position vector , x: momentum vector
For any C^∞-functions $f = f(y,x), g = g(y,x) \in C^\infty(\mathbf{k}^2)$,

$$\begin{aligned} \{f,g\} &= \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} - \frac{\partial g}{\partial y}\frac{\partial f}{\partial x} \\ &= \left(\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x}\right)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial x} \end{pmatrix} \\ &= \bigtriangledown f \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\bigtriangledown g)^t \end{aligned}$$

In particular, in the polynomial ring $\mathbf{k}[y,x] \subset C^\infty(\mathbf{k}^2)$

$$\{y,x\} = 1$$

Poisson algebra 2

All vector spaces are over a field $\mathbf{k}$ of characteristic zero.

A vector space $R = (R, \cdot, \{-,-\})$ is said to be a *Poisson algebra* if

- $(R, \cdot)$ is a commutative algebra.
- $(R, \{-,-\})$ is a Lie algebra.
- $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in R$. (Leibniz rule)

Example [Jordan and Oh, 2012]

- Let $R = \mathbb{C}[x, y, z]$.

- Fix $s, t \in R \setminus \{0\}$ such that s and t are coprime.

For $f, g \in R$, define

$$\{f, g\} = t^2 \begin{vmatrix} \nabla f \\ \nabla g \\ \nabla(s/t) \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ t\frac{\partial s}{\partial x} - s\frac{\partial t}{\partial x} & t\frac{\partial s}{\partial y} - s\frac{\partial t}{\partial y} & t\frac{\partial s}{\partial z} - s\frac{\partial t}{\partial z} \end{vmatrix}$$

- $\{x, y\} = t\frac{\partial s}{\partial z} - s\frac{\partial t}{\partial z}$, $\{y, z\} = t\frac{\partial s}{\partial x} - s\frac{\partial t}{\partial x}$, $\{z, x\} = t\frac{\partial s}{\partial y} - s\frac{\partial t}{\partial y}$.

Poisson primitive ideal [Oh, 1999]

R: Poisson algebra

• A Poisson ideal I of R is said to be a *Poisson primitive ideal* if there exists a maximal ideal M of R such that I is the largest Poisson ideal contained in M.
That is, I is the sum of all Poisson ideals contained in M.

Let B be an associative algebra.
• An ideal I of B is a (left) *primitive ideal* if and only if there exists a maximal left ideal M of B such that I is the largest ideal contained in M.

• Poisson primitive ideal $\Rightarrow$ Poisson prime ideal.

Poisson module [Oh, 1999]

A vector space M is said to be a *Poisson R-module* if

• there exists a bilinear map

$$R \times M \to M, \quad (a, m) \mapsto am$$

makes M into a left $(R, \cdot)$-module,

• there exists a bilinear map

$$R \times M \to M, \quad (a, m) \mapsto \{a, m\}$$

makes M into a left $(R, \{-,-\})$-Lie module,

for $a, b \in R$ and $m \in M$,
• $\{a, b\}m = \{a, bm\} - b\{a, m\}$,
• $\{ab, m\} = a\{b, m\} + b\{a, m\}$.

Poisson enveloping algebra [Oh, 1999]

Let M be a Poisson module. Then

- there exists an algebra homomorphism

$$\alpha : R \to \mathrm{End}_{\mathbf{k}}(M), \quad \alpha(a)m = am.$$

- there exists a Lie homomorphism

$$\beta : (R, \{-,-\}) \to \mathrm{End}_{\mathbf{k}}(M)_L, \quad \beta(a)m = \{a, m\}.$$

For $a, b \in R$,
- $\alpha(\{a, b\}) = \beta(a)\alpha(b) - \alpha(b)\beta(a) = [\beta(a), \alpha(a)]$.
- $\beta(ab) = \alpha(a)\beta(b) + \alpha(b)\beta(a)$.

Concentrate on the triple $(\mathrm{End}_{\mathbf{k}}(M), \alpha, \beta)$.

Poisson enveloping algebra

Let $U(R)$ be an associative algebra,
$\alpha : R \to U(R)$ an algebra homomorphism,
$\beta : (R, \{-,-\}) \to U(R)_L$ a Lie homomorphism.

The triple $(U(R), \alpha, \beta)$ is said to be a *Poisson enveloping algebra* of R if,
for any triple (B, γ, δ) such that
$\gamma(\{a, b\}) = [\delta(a), \gamma(a)]$ and
$\delta(ab) = \gamma(a)\delta(b) + \gamma(b)\delta(a)$,
there exists a unique algebra homomorphism h such that
$h\alpha = \gamma, \quad h\beta = \delta.$

$$\begin{tikzcd} R \arrow[r, "\alpha{,}\beta"] \arrow[d, "\gamma{,}\delta"'] & U(R) \arrow[dl, dotted, "h"] \\ B & \end{tikzcd}$$

Theorem [Oh, 1999]

Let $U(R)$ be the Poisson enveloping algebra of R.
Then, for any vector space M,

M is a Poisson R-module if and only if M is a $U(R)$-module.

Corollary

M is a simple Poisson R-module if and only if M is a simple $U(R)$-module.

Motivation

For an ideal I of an associative algebra B,
I is a primitive ideal if and only if $I = \mathrm{ann}_B(M)$ for some simple B-module.

Simple Poisson module [Oh, Sim and Zhu, 2019]

• There exits a simple Poisson R-module M such that $\text{ann}_R(M)$ is not Poisson primitive.

• Let R be an affine Poisson algebra. Then
Every Poisson primitive ideal is an annihilator of simple Poisson module.

• Let R be an affine Poisson algebra over $\mathbb{C}$. Then
Every annihilator of simple Poisson module is Poisson primitive. [Launois, etc., 2018]

Quantization (Star product)

• R a Poison algebra
• A $\mathbf{k}[[t]]$-algebra $Q = R[[t]]$ with star product $*$ such that for any $a, b \in R \subseteq Q$,

$$a * b = ab + B_1(a,b)t + B_2(a,b)t^2 + \dots$$

subject to

$$\{a, b\} = t^{-1}(a * b - b * a)|_{t=0}, \qquad \cdots \quad (\ddagger)$$

where $B_i : R \times R \longrightarrow R$ are bilinear products,
is called a *quantization* of R.

Weyl algebra 1

• The Poisson Weyl algebra is the **k**-algebra $R = \mathbf{k}[x, y]$ with Poisson bracket

$$\{f, g\} = -\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} + \frac{\partial g}{\partial x}\frac{\partial f}{\partial y}$$

for all $f, g \in R$, namely $\{y, x\} = 1$.

• $R[[t]] = \mathbf{k}[x, y][[t]]$ the formal power series over R
The Moyal-Weyl quantization is the $\mathbf{k}[[t]]$-algebra $R[[t]] = \mathbf{k}[x, y][[t]]$ subject to the relation

$$yx = xy + t$$

• The Weyl algebra W is the **k**-algebra generated by x, y subject to the relation

$$yx = xy + 1$$

Weyl algebra 2

• The Weyl algebra W is the **k**-algebra generated by x, y subject to the relation

$$yx = xy + 1$$

That is,

$$W = R[[t]]|_{t=1} = R[[t]]/(t-1)R[[t]] \qquad (???)$$

• For

$$f = 1 + tx + t^2 + t^3 + \ldots \in R[[t]] = \mathbf{k}[x, y][[t]],$$

$$f|_{t=1} = 1 + x + 1^2 + 1^3 + \ldots \qquad (???)$$

Semiclassical limit 1

B: an associative algebra with $t \in B$ such that

- t: a nonzero, nonunit, non-zero-divisor, central element,
- B/tB is commutative.

Then $\overline{B} := B/tB$ is a Poisson algebra with

$$\{\overline{a}, \overline{b}\} = \overline{t^{-1}(ab - ba)}$$

for all $\overline{a} = a + tB, \overline{b} = b + tB \in \overline{B}$.

Semiclassical limit 2

- B: a *quantization* of $\overline{B}$,
- $\overline{B} = B/tB$: a *semiclassical limit* of B,
- $B_\lambda := B/(t-\lambda)B$: a *deformation* of $\overline{B}$ (or B),

where $0 \neq \lambda \in \mathbf{k}$ such that $t - \lambda$ is a nonzero and nonunit in B.

Example 1

- Let B be the $\mathbf{k}[t]$-algebra generated by x, y subject to the relation

$$yx - xy - t.$$

$$B := \mathbf{k}[t] < x, y > /(yx - xy - t)$$

- Semiclassical limit.

$$\overline{B} = B/tB \cong \mathbf{k}[x, y], \quad \{y, x\} = 1$$

- Deformation.

$$B_1 := B/(t-1)B = W$$

In general, for each $0 \neq \lambda \in \mathbf{k}$,

$$B_\lambda := B/(t-\lambda)B = \mathbf{k}[x, y], \quad yx = xy + \lambda$$

Example 2

• Quantized algebra.

Let $\mathbb{F} = \mathbf{k}[(t+1)^{\pm 1}]$ and let

$$B := \mathbb{F} < x, y > /(yx - (t+1)xy)$$

• Semiclassical limit.

$$\overline{B} = B/tB \cong \mathbf{k}[x, y], \quad \{y, x\} = xy$$

• Deformation.

Let $q \in \mathbf{k}$ be not a root of unity and let

$$\mathcal{O}_q(\mathbf{k}^2) := B/(t+1-q)B$$

That is,

$$\mathcal{O}_q(\mathbf{k}^2) = \mathbf{k}[x, y], \quad yx = qxy,$$

called a *coordinate ring of quantized affine 2-space*.

Continued

$$\mathrm{Prim}(\mathcal{O}_q(\mathbf{k}^2)) = \{(0), (x-\alpha, y), (x, y-\beta) \mid \alpha, \beta \in \mathbf{k}\}$$

$$\mathrm{P.Prim}(\overline{B}) = \{(0), (x-\alpha, y), (x, y-\beta) \mid \alpha, \beta \in \mathbf{k}\}$$

• This phenomenon holds for many quantized algebras and their semiclassical limits.

Homomorphism [Myung and Oh, 2019]

• A natural group homomorphism

$$\varphi : \mathrm{Aut}(B) \to \mathrm{P.Aut}(\overline{B})$$

is constructed, where B is a quantized algebra, and $\overline{B}$ is its semiclassical limit.

• Kontsevich's conjecture (2005):
The group of automorphisms of Weyl algebra W_n is isomorphic to the group of Poisson Weyl algebras, which was solved by Alexei Kanel-Belov, Andrey Elishev and Jie-Tai Yu in December of 2018.

Dixmier's conjecture

• Dixmier's conjecture [1968] : Every endomorphism of the Weyl algebra W_n is an automorphism.

• Jacobian conjecture [Keller, 1939] : Let $f = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function such that $\det(\frac{\partial f_j}{\partial x_i})_{i,j=1}^{n}$ is nonzero for every point then f is invertible and its inverse f^{-1} is also a polynomial function.

• Kanel-Belov and Kontsevich [2007] proved that Dixmier' conjecture is equivalent to Jacobian conjecture.

• Conjecture: Every Poisson endomorphism of the Poisson Weyl algebra $\overline{B}_n$ is a Poisson automorphism, where $\overline{B}_n$ is a semiclassical limit of W_n.

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Thank you for your attention