Density of g-vector cones from triangulated surfaces

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Today's talk

1. Notations
2. τ-tilting theory
3. g-vectors and g-vector cones
4. Triangulated surfaces
5. Result

Notations

- A : a finite dimensional algebra over a field k.
- $\operatorname{mod} A$: the category of finitely generated right A-modules.
- τ : the Auslander-Reiten translation of $\operatorname{mod} A$.

- $|M|$: the number of non-isomorphic indecomposable direct summands of $M \in \operatorname{mod} A$.
- $n = |A|$.

τ-tilting pairs

Definition

Let $M \in \operatorname{mod} A$ and P a projective A-module. We say that (M, P) is

- a τ-rigid pair if $\operatorname{Hom}_A(M, \tau M) = 0$ and $\operatorname{Hom}_A(P, M) = 0$.
- a τ-tilting pair if it is a τ-rigid pair and $|M| + |P| = n$.
- an almost τ-tilting pair if it is a τ-rigid pair and $|M| + |P| = n - 1$.

- $\tau\text{-tilt}\, A := \{$(basic) τ-tilting pairs$\}$.
- $\mathrm{irigid}\, A := \{$indecomposable τ-rigid pairs$\}$
  $= \{$indec. direct summands of τ-tilting pairs in $\tau\text{-tilt}\, J_T\}$

Remark

$\{$support τ-tilting modules$\} \ni M \leftrightarrow (M, P) \in \tau\text{-tilt}\, A$

Mutation

Theorem (Adachi-Iyama-Reiten, 2014)

Any almost τ-tilting pair is a direct summand of exactly two τ-tilting pairs.

For $U = \bigoplus_{i=1}^{n} U_i \in \tau\text{-tilt}\, A$ and $j \in \{1, \dots, n\}$, there is a unique $U'_j \in \operatorname{irigid} A$ such that $U'_j \neq U_j$ and $U' := \bigoplus_{i \neq j} U_i \oplus U'_j \in \tau\text{-tilt}\, A$.

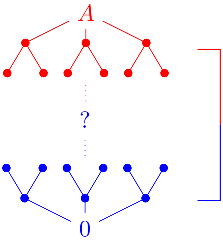
$$U \xleftrightarrow[\text{at } j]{\text{mutation}} U'$$

$$\tau\text{-tilt}^{\pm} A := \left\{ \begin{array}{l} \tau\text{-tilting pairs obtained from } (A, 0) \text{ or } (0, A) \\ \text{by a sequence of mutations} \end{array} \right\} \subseteq \tau\text{-tilt}\, A.$$

$$\operatorname{irigid}^{\pm} A := \left\{ \begin{array}{l} \text{indec. direct summands of } \tau\text{-tilting pairs} \\ \text{in } \tau\text{-tilt}^{\pm} A \end{array} \right\} \subseteq \operatorname{irigid} A.$$

Question

The exchange graph of $\tau\text{-tilt}\, A$



$\tau\text{-tilt}^{\pm} A$

Question

$\tau\text{-tilt}\, A \setminus \tau\text{-tilt}^{\pm} A =$?.

We only consider finite dimensional algebras defined from triangulated surface. Before we introduce them, we study a useful tool "g-vector cones".

g-vectors of τ-rigid pairs

Fix $A = \bigoplus_{i=1}^{n} P_i$, where P_i is an indecomposable projective A-module.

Definition

Let $M \in \operatorname{mod} A$. There is a minimal projective presentation of M

$$\bigoplus_{i=1}^{n} P_i^{m_i'} \to \bigoplus_{i=1}^{n} P_i^{m_i} \to M \to 0.$$

The g-vector of M is a vector

$$g^M = (m_1 - m_1', \ldots, m_n - m_n') \in \mathbb{Z}^n.$$

For a τ-rigid pair (M, P), its g-vector is $g^{(M,P)} := g^M - g^P$.

Theorem (AIR)

For $\bigoplus_{i=1}^{n} U_i \in \tau\text{-tilt}\, A$, then $g^{U_1}, \ldots, g^{U_n}$ form a basis of $\mathbb{R}^n$.

g-vector cones of τ-tilting pairs

Definition

To $U = \bigoplus_{i=1}^{m} U_i \in \tau\text{-tilt}\, A$, we associate the g-vector cone

$$C(U) := \Big\{ \sum_{i=i}^{m} a_i g^{U_i} \mid a_i \geq 0 \Big\}.$$

Theorem (Reading, 2014)

If A is τ-tilting finite, that is $\#\, \tau\text{-tilt}\, A < \infty$, then we have

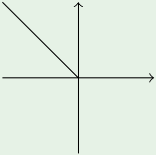
$$\bigcup_{U \in \tau\text{-tilt}\, A} C(U) = \mathbb{R}^n .$$

Example (Type A_2)

For a quiver $Q = (1 \leftarrow 2)$, the AR quiver of the path algebra kQ is

$$\begin{tikzcd}[column sep=small] & \binom{0}{1} \arrow[dr] & \\ \binom{1}{0} \arrow[ur] & & \binom{-1}{1} \end{tikzcd} \quad \text{and} \quad \tau\text{-tilt}\, kQ = \left\{ \begin{matrix} \left(\binom{1}{0} \oplus \binom{0}{1}, 0\right), & \left(\binom{-1}{1} \oplus \binom{0}{1}, 0\right) \\ \left(\binom{1}{0}, \binom{0}{1}\right), & \left(\binom{-1}{1}, \binom{1}{0}\right) \\ \left(0, \binom{1}{1} \oplus \binom{0}{1}\right) & \end{matrix} \right\},$$

where the modules are described by their g-vectors. Then their g-vector cones span the ambient space $\mathbb{R}^2$ as the right diagram.

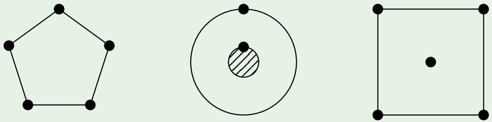


In this talk, we consider an analogue of the theorem for finite dimensional algebras defined from triangulated surfaces.

Triangulated surfaces

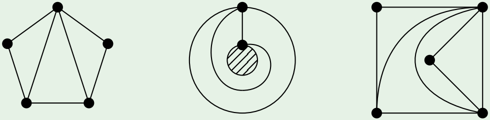
- S : a connected compact oriented Riemann surface
- M : a finite set of marked points on S with at least one marked point on each component of the boundary of S

Example



An arc of (S, M) is a non-self-crossing curve (considered up to homotopy) with endpoints in M.
A triangulation of (S, M) is a maximal set of pairwise non-crossing arcs.

Example



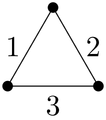
In general, we use tagged arcs and tagged triangulations instead of arcs and triangulations.

Jacobian algebras

To a tagged triangulation T of (S, M), Labardini-Fragoso associated the finite dimensional Jacobian algebra J_T.

Roughly speaking : Let Q_T be a quiver whose

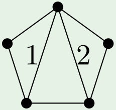
- vertices are arcs of T,
- arrows are obtained as follows:



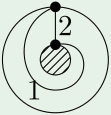
$$\rightarrow \quad \begin{tikzcd} 1 \arrow[dr] & & 2 \arrow[ll] \\ & 3 \arrow[ur] & \end{tikzcd}$$

$$J_T = kQ_T / \langle \text{paths of length } 2 \text{ in each triangle} \rangle.$$

Example



$Q_T = (1 \leftarrow 2)$



$Q_T = (1 \leftleftarrows 2)$

These algebras have the following property.

Theorem (AIR, Fomin-Shapiro-Thurston, FT, Fu-Keller)

For a tagged triangulation T of (S,M), there is a bijection

$$\mathrm{irigid}^{\pm} J_T \longleftrightarrow \{\textit{Tagged arcs of } (S,M)\}.$$

It induces a bijection

$$\tau\text{-}\mathrm{tilt}^{\pm} J_T \longleftrightarrow \{\textit{Tagged triangulations of } (S,M)\}$$

which commutes with mutations and flips.

Result

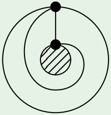
Theorem

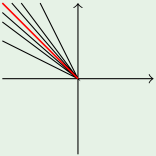
For a tagged triangulation T of (S, M), we have

$$\overline{\bigcup_{U \in \tau\text{-tilt}^{\pm} J_T} C(U)} = \mathbb{R}^n .$$

To prove this theorem, we study geometric tools (laminations, their shear coordinates, Dehn twists, and so on) and apply them to representation theory via the above bijections.

Example

Let $T =$ . Then $Q_T = (1 \leftleftarrows 2)$. In this case,



there are infinitely many τ-tilting pairs whose g-vector cones are as in the right diagram:

where no g-vector cones contain the red ray. However, there are infinitely many g-vector cones which are getting closer to it. Therefore,

$$\bigcup_{U \in \tau\text{-tilt}\, kQ_T} C(U) \neq \mathbb{R}^2 \quad \text{and} \quad \overline{\bigcup_{U \in \tau\text{-tilt}\, kQ_T} C(U)} = \mathbb{R}^2 \, .$$

Theorem (Demonet-Iyama-Jasso, 2019)

For $U, V \in \tau\text{-tilt}\, A$*, if* $C(U) \cap C(V)$ *has dimension* n*, then* $U = V$*.*

Corollary

For a tagged triangulation T *of* (S, M)*, we have*

$$\tau\text{-tilt}\, J_T = \tau\text{-tilt}^{\pm}\, J_T.$$

Corollary

For a tagged triangulation T *of* (S, M)*, there is a bijections*

$$\begin{aligned} \mathrm{irigid}\, J_T \quad &\longleftrightarrow \quad \{\textit{Tagged arcs of } (S, M)\} \\ \tau\text{-tilt}\, J_T \quad &\longleftrightarrow \quad \{\textit{Tagged triangulations of } (S, M)\}. \end{aligned}$$

Reference

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