Semibricks, wide subcategories and recollements

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1. Semibricks.etc

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Semibricks

1. A module $S \in \operatorname{mod} A$ is called a *brick* if $\operatorname{End}_A(S)$ is a division algebra (i.e., the non-trivial endomorphisms are invertible).
   $\operatorname{brick} A = \{\text{isoclasses of bricks in } \operatorname{mod} A\}$.
2. A set of $\mathcal{S} \in \operatorname{mod} A$ of isoclasses of bricks is called a *semibrick* if $\operatorname{Hom}_A(S_1, S_2) = 0$ for any $S_1 \neq S_2 \in \mathcal{S}$.
   $\operatorname{sbrick} A = \{\text{semibricks in } \operatorname{mod} A\}$.

Wide subcategories(Hov)

A full subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is called *wide* if it is abelian and closed under extensions.

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Put

$$\text{wide}\mathcal{A} = \{\text{wide subcategories of } \mathcal{A}\}.$$
$$\text{wide}A = \{\text{wide subcategories of } \operatorname{mod} A\}.$$
$$\text{wide}_{\mathcal{C}}\mathcal{A} = \{\text{wide subcategories of } \mathcal{A} \text{ containing } \mathcal{C}\}.$$

Support τ-tilting modules (Adachi-Iyama-Reiten)

Let (X,P) be a pair with $X \in \mathrm{mod}\, A$ and $P \in \mathrm{proj}\, A$. We call (X,P) a **support τ-tilting pair** if

1. X is **τ-rigid**, i.e., $\mathrm{Hom}_A(X, \tau X)=0$
2. $\mathrm{Hom}_A(P, X)=0$
3. $|X| + |P| = |A|$

In this case, X is called a **support τ-tilting module**.

Put

$$\mathrm{s}\tau\text{-tilt}A = \{\text{basic support } \tau\text{-tilting } A\text{-modules}\}.$$

Related works

1. Representations of K-species and bimodules. (Rin,1976)
2. τ-tilting theory. (AIR,2014)
3. τ-tilting finite algebras, g-vectors and brick-τ-rigid correspondence. (DIJ,2019)
4. Semibricks. (As,2019)

Recollements(BBD,FP,Ha,K)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Then a recollement of $\mathcal{B}$ relative to $\mathcal{A}$ and $\mathcal{C}$, diagrammatically expressed by

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, tail, "i_*" description] & & \mathcal{B} \arrow[ll, two heads, bend right, "i^*"'] \arrow[ll, two heads, bend left, "i^!"] \arrow[rr, two heads, "j^*" description] & & \mathcal{C} \arrow[ll, tail, bend right, "j_!"'] \arrow[ll, tail, bend left, "j_*"]
\end{tikzcd}$$

which satisfy the following three conditions:

1. (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$ and (j^*, j_*) are adjoint pairs;
2. $i_*, j_!$ and j_* are fully faithful functors;
3. $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

Remark

1. i_* and j^* are both right adjoint functors and left adjoint functors, therefore they are exact functors of abelian categories.
2. $i^*i_* \cong id, i^!i_* \cong id, j^*j_! \cong id$ and $j^*j_* \cong id$. Also $i^*j_! = 0, i^!j_* = 0$.
3. Denote by $R(\mathcal{A}, \mathcal{B}, \mathcal{C})$ a recollement of $\mathcal{B}$ relative to $\mathcal{A}$ and $\mathcal{C}$ as above and $R(A, B, C)$ a recollement of $\operatorname{mod} B$ relative to $\operatorname{mod} A$ and $\operatorname{mod} C$.

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Associated to a recollement there is a seventh funtor $j_{!*} := Im(j_! \to j_*) : \operatorname{mod} C \to \operatorname{mod} B$ called the intermediate extension functor.

Intermediate extension functor

1. $i^* j_{!*} = 0, i^! j_{!*} = 0$.
2. $j^* j_{!*} \cong id$ and the functors $i_*, j_!, j_*$ and $j_{!*}$ are full embeddings.
3. The functor $j_{!*}$ sends simples in $\mathrm{mod}\, C$ to simples in $\mathrm{mod}\, B$. There is a bijection between sets of isomorphism classes of simples: (gluing simple modules)

$$\{\text{simples} \in \mathrm{mod}\, A\} \sqcup \{\text{simples} \in \mathrm{mod}\, C\} \to \{\text{simples} \in \mathrm{mod}\, B\}$$

given by mapping a simple $M_L \in \mathrm{mod}\, A$ to $i_*(M_L)$ and a simple $M_R \in \mathrm{mod}\, C$ to $j_{!*}(M_R)$.

Related works

1. Analysis and topology on singular spaces. (BBD,1981)
2. Recollements of extension algebras. (CL,2003)
3. One-point extension and recollement. (LL,2008)
4. From recollement of triangulated categories to recollement of abelian categories. (LW,2010)
5. Weight structures vs. t-structures; weight filtrations, spectral sequences, and complexes. (B,2010)
6. Gluing silting objects. (LVY,2014)
7. Lifting of recollements and gluing of partial silting sets. (SZ,arXiv2018)

Connection(Rin,Asai)

Bijections:

Ringel's bijection: sbrick$A \longrightarrow$wide$A$

Asai's bijection: sτ-tilt$A \longrightarrow f_L-$sbrick$A$ via $M \longrightarrow$ind$(M/\text{rad}_B M)$

If A is τ-tilting finite, f_L-sbrick$A =$sbrick$A$ and there is a bijection sτ-tilt$A \longrightarrow$sbrick$A$

2. Gluing semibricks

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Lemma

If $F : \mathrm{mod}\, A \to \mathrm{mod}\, B$ is a fully faithful functor, then we have $F(\mathrm{brick} A) \subseteq \mathrm{brick} B$ and $F(\mathrm{sbrick} A) \subseteq \mathrm{sbrick} B$.

Proposition

Let $R(A, B, C)$ be a recollement.

1. $i_*(\mathrm{brick} A) \subseteq \mathrm{brick} B$ and $i_*(\mathrm{sbrick} A) \subseteq \mathrm{sbrick} B$;
2. $j_!(\mathrm{brick} C) \subseteq \mathrm{brick} B$ and $j_!(\mathrm{sbrick} C) \subseteq \mathrm{sbrick} B$;
3. $j_*(\mathrm{brick} C) \subseteq \mathrm{brick} B$ and $j_*(\mathrm{sbrick} C) \subseteq \mathrm{sbrick} B$;
4. $j_{!*}(\mathrm{brick} C) \subseteq \mathrm{brick} B$ and $j_{!*}(\mathrm{sbrick} C) \subseteq \mathrm{sbrick} B$.

Theorem{Gluing semibricks}

Let $R(A, B, C)$ be a recollement.
$i_*(\text{sbrick}A) \sqcup j_{!*}(\text{sbrick}C) \subseteq \text{sbrick}B$.

There is an injection between sets of isomorphism classes of semibricks:

$$\text{sbrick}A \sqcup \text{sbrick}C \to \text{sbrick}B$$

through a semibrick $\mathcal{S}_L \in \operatorname{mod} A$ and a semibrick $\mathcal{S}_R \in \operatorname{mod} C$ into $i_*(\mathcal{S}_L) \sqcup j_{!*}(\mathcal{S}_R)$.

Theorem

Let $R(A, B, C)$ be a recollement. If B is τ-tilting finite, A and C are τ-tilting finite.

Corollary

Let A be a finite dimensional algebra and e an idempotent element of A. If A is τ-tilting finite, it follows that eAe and $A/\langle e \rangle$ are τ-tilting finite.

Let $R(A, B, C)$ be a recollement of module categories and B a τ-tilting finite algebra. Since semibricks can be glued via a recollement, the natural question is the following:

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Question

Given a recollement of module categories, support τ-tilting modules M_A and M_C in $\mathrm{mod}\, A$ and $\mathrm{mod}\, C$, is it possible to construct a support τ-tilting module in $\mathrm{mod}\, B$ corresponding to the glued semibrick?

Let $R(A, B, C)$ be a recollement of module categories and B a τ-tilting finite algebra. Since semibricks can be glued via a recollement, the natural question is the following:

Question

Given a recollement of module categories, support τ-tilting modules M_A and M_C in $\operatorname{mod} A$ and $\operatorname{mod} C$, is it possible to construct a support τ-tilting module in $\operatorname{mod} B$ corresponding to the glued semibrick?

Answer

Yes, there exists a unique support τ-tilting B-module M_B which is associated with the induced semibrick $i_*(\mathcal{S}_A) \sqcup j_{!*}(\mathcal{S}_C)$.

Example {Gluing support τ-tilting modules over τ-tilting finite algebras}

Let A be the path algebra over a field of the quiver $1 \to 2 \to 3$. If e is the idempotent $e_1 + e_2$, then as a right A-module $A/\langle e\rangle$ is isomorphic to S_3 and eAe is the path algebra of the quiver $1 \to 2$. In this case, there is a recollement as follows:

$$\begin{tikzcd}[column sep=huge]
\mathrm{mod}\,(A/\langle e\rangle) \arrow[r, tail, "i_*" description] & \mathrm{mod}\,A \arrow[l, two heads, bend right, "i^*"'] \arrow[l, two heads, bend left, "i^!"] \arrow[r, two heads, "j^*" description] & \mathrm{mod}\,(eAe) \arrow[l, tail, bend right, "j_!"'] \arrow[l, tail, bend left, "j_*"]
\end{tikzcd}$$

where $i^* = - \otimes_A A/\langle e\rangle$, $j_! = - \otimes_{eAe} eA$, $i^! = \mathrm{Hom}_A(A/\langle e\rangle, -)$, $i_* = - \otimes_{A/\langle e\rangle} A/\langle e\rangle$, $j^* = - \otimes_A Ae$, $j_* = \mathrm{Hom}_{eAe}(Ae, -)$.

Table:

$s\tau$-tilt $(A/\langle e \rangle)$	$s\tau$-tilt A	$s\tau$-tilt (eAe)
3	$3 \ \begin{smallmatrix}2\\3\end{smallmatrix} \ \begin{smallmatrix}1\\2\\3\end{smallmatrix}$	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 2$
3	$3 \ \begin{smallmatrix}1\\2\\3\end{smallmatrix} \ 1$	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 1$
3	$3 \ \begin{smallmatrix}2\\3\end{smallmatrix}$	2
3	$3 \ 1$	1
3	3	0
0	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 2$	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 2$
0	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 1$	$\begin{smallmatrix}1\\2\end{smallmatrix} \ 1$
0	2	2
0	1	1
0	0	0

Example

Let A be the preprojective algebra of type A_3 which is given by the following quiver and relation $aa' = 0, b'b = 0, bb' = a'a$.

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "a'"] \arrow[r, bend left, "b"] & 3 \arrow[l, bend left, "b'"] \end{tikzcd}$$

Let $e = e_1 + e_3$. Then as a right A-module $A/\langle e \rangle$ is isomorphic to S_2 and eAe is the preprojective algebra of type A_2. Then there is a recollement $R(A/\langle e \rangle, A, eAe)$ induced by the idempotent e.

sτ-tilt $(A/\langle e\rangle)$	sτ-tilt A	sτ-tilt (eAe)
2	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}2\\1\,3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\ \begin{smallmatrix}3\\1\end{smallmatrix}$
2	$\begin{smallmatrix}2\\3\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}$	$3\ \begin{smallmatrix}3\\1\end{smallmatrix}$
2	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}$	$\begin{smallmatrix}1\\3\end{smallmatrix}\ 1$
2	$\begin{smallmatrix}2\\3\end{smallmatrix}\ \begin{smallmatrix}3\\2\end{smallmatrix}$	3
2	$\begin{smallmatrix}1\\2\end{smallmatrix}\ \begin{smallmatrix}2\\1\end{smallmatrix}$	1
2	2	0
0	$3\ 1$	$\begin{smallmatrix}1\\3\end{smallmatrix}\ \begin{smallmatrix}3\\1\end{smallmatrix}$
0	$3\ \begin{smallmatrix}3\\2\end{smallmatrix}\ \begin{smallmatrix}3\\2\\1\end{smallmatrix}$	$3\ \begin{smallmatrix}3\\1\end{smallmatrix}$
0	$\begin{smallmatrix}1\\2\\3\end{smallmatrix}\ \begin{smallmatrix}1\\2\end{smallmatrix}\ 1$	$\begin{smallmatrix}1\\3\end{smallmatrix}\ 1$
0	3	3
0	1	1
0	0	0

3. Reduction of wide subcategories

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Theorem

There is a bijection

$$\mathrm{wide}_{i_*(\mathcal{A})}\mathcal{B} \leftrightarrow \mathrm{wide}\mathcal{C}$$

given by $\mathrm{wide}_{i_*(\mathcal{A})}\mathcal{B} \ni \mathcal{C} \mapsto j^*(\mathcal{C}) \in \mathrm{wide}\mathcal{C}$ and $\mathrm{wide}\mathcal{C} \ni \mathcal{W} \mapsto \mathcal{C} = \{M \in \mathcal{B} | j^*(M) \in \mathcal{W}\} \in \mathrm{wide}_{i_*(\mathcal{A})}\mathcal{B}$.

Theorem

If $\mathcal{C} \subset \mathcal{B}$ is wide and satisfies $i_*(\mathcal{A}) \subset \mathcal{C}$, then we can get a recollement of wide subcategories as follows:

$$\begin{tikzcd}[column sep=huge]
\mathcal{A} \arrow[r, tail, "\overline{i_*}" description] & \mathcal{C} \arrow[l, two heads, shift right=3ex, "\overline{i^*}"'] \arrow[l, two heads, shift left=3ex, "\overline{i^!}"] \arrow[r, two heads, "\overline{j^*}" description] & j^*(\mathcal{C}) \arrow[l, tail, shift right=3ex, "\overline{j_!}"'] \arrow[l, tail, shift left=3ex, "\overline{j_*}"]
\end{tikzcd} .$$

Example

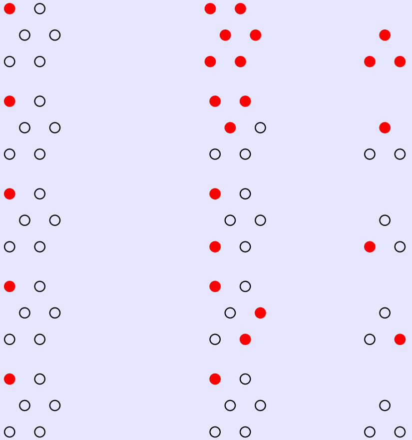
Let A be the path algebra over a field of the quiver $1 \leftarrow 2 \rightarrow 3$, of type A_3. If e is the idempotent $e_2 + e_3$, then as a right A-module A/AeA is isomorphic to S_1. In this case, there is a recollement as follows:

$$\begin{tikzcd}[column sep=huge] \operatorname{mod}(A/AeA) \arrow[r, tail, "i_*" description] & \operatorname{mod} A \arrow[l, two heads, bend right, "i^*"'] \arrow[l, two heads, bend left, "i^!"] \arrow[r, two heads, "j^*" description] & \operatorname{mod}(eAe) \arrow[l, tail, bend right, "j_!"'] \arrow[l, tail, bend left, "j_*"] \end{tikzcd}$$

where $i^* = - \otimes_A A/\langle e \rangle$, $j_! = - \otimes_{eAe} eA$, $i^! = \mathrm{Hom}_A(A/\langle e \rangle, -)$, $i_* = - \otimes_{A/\langle e \rangle} A/\langle e \rangle$, $j^* = - \otimes_A Ae$, $j_* = \mathrm{Hom}_{eAe}(Ae, -)$.

Table:

$\operatorname{mod}(A/AeA)$	$\mathcal{C}$	$j^*(\mathcal{C})$



Thanks for your attention!