The 53rd Symposium on Ring Theory

and Representation Theory

ABSTRACT

Yamaguchi University, Yamaguchi (Zoom, Online)

September 6 – 9, 2021

Program

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Vanishing of Ext modules over Cohen-Macaulay rings

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Large tilting objects induced by codimension functions and homomorphic images of Cohen-Macaulay rings

11:20–11:50 Yasuaki Gyoda (Nagoya University)

Positive cluster complexes and τ-tilting simplicial complexes

13:30–14:00 Takahiro Honma (Tokyo University of Science), Takuma Aihara (Tokyo University of Science)

τ-tilting finite triangular matrix algebras

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On almost N-projective modules and generalized N-projective modules

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Subcategories and silting objects of Noetherian algebras

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Structure theorem for flat cotorsion modules over Noether algebras

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On τ-tilting finiteness of tensor product algebras between simply connected algebras

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Ryotaro Koshio (Tokyo University of Science), Yuta Kozakai (Tokyo University of Science)
On support τ-tilting modules and semibricks for blocks of group algebras

Tomohiro Itagaki (Takasaki City University of Economics), Kazunori Nakamoto (University of Yamanashi), Takeshi Torii (Okayama University)
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On two-sided Harada rings

Naoya Hiramatsu (National Institute of Technology, Kure College)
Krull–Gabriel dimension of Cohen–Macaulay modules over hypersurfaces of type (A_∞)

Yasuaki Hiraoka (Osaka City University), Yuichi Ike (Kyoto University), Michio Yoshiwaki (University of Tokyo)
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Vanishing of Ext modules over Cohen-Macaulay rings

Kaito Kimura, Yuya Otake, Ryo Takahashi

Auslander and Reiten [4] proposed the *generalized Nakayama conjecture*, which is rooted in the *Nakayama conjecture* [10]. In addition, they proposed another conjecture about projectivity of module by vanishing of Ext modules, which is called the *Auslander-Reiten conjecture*, and proved that this conjecture is true if and only if the generalized Nakayama conjecture is true. This long-standing conjecture is known to be true over several classes of algebras. For example, Auslander and Reiten [4] proved the conjecture for algebras of finite representation type. Hoshino [7] proved that it holds for symmetric Artin algebras with radical cube zero. The Auslander-Reiten conjecture is closely related to other important conjectures such as the Tachikawa conjecture [12].

The Auslander-Reiten conjecture remains meaningful for arbitrary commutative noetherian rings for formalization by Auslander, Ding, and Solberg [3]:

Conjecture 1.1 (Auslander-Reiten)**.** *Let R be a commutative noetherian ring and let M be a finitely generated R-module. If $\mathrm{Ext}^i_R(M,M) = \mathrm{Ext}^i_R(M,R) = 0$ for all $i \geq 1$, then M is projective.*

Conjecture 1.1 is known to hold for complete intersection local rings [3], locally excellent Cohen-Macaulay normal rings containing the field of rational numbers [8], and Gorenstein normal rings [1]. As a subject related to Conjecture 1.1, a lot of criteria for a given module to be projective have been described in terms of vanishing of Ext modules so far; see [1, 2, 3, 5, 6, 8, 11] for instance. In this talk, we consider the above problems over Cohen-Macaulay rings, which form one of the most important classes in commutative algebra.

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Large tilting objects induced by codimension functions and homomorphic images of Cohen-Macaulay rings

Tsutomu Nakamura

This talk is based on joint work with Michal Hrbek and Jan Šťovíček.

Let R be a commutative noetherian ring. Denote by $D(R)$ the unbounded derived category of R. For each $\mathfrak{p} \in \operatorname{Spec} R$, denote by $\Gamma_{\mathfrak{p}}$ the $\mathfrak{p}$-torsion functor $\varinjlim_{n\geq 1} \operatorname{Hom}_R(R/\mathfrak{p}^n, -) : \operatorname{Mod} R \to \operatorname{Mod} R$. We will report the following result, which explicitly gives a lot of (possibly non-compact) silting objects by a consistent way.

Theorem 1. *Let* $\mathsf{f} : \operatorname{Spec} R \to \mathbb{Z}$ *be a strictly increasing function. Then*

$$\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\mathsf{f}(\mathfrak{p})} \mathbf{R}\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}$$

is a silting object in $D(R)$.

For each $\mathfrak{p} \in \operatorname{Spec} R$, denote by $\widehat{R_{\mathfrak{p}}}$ the $\mathfrak{p}$-adic completion of $R_{\mathfrak{p}}$. Then $\widehat{R_{\mathfrak{p}}}$ admits a dualizing complex $D_{\widehat{R_{\mathfrak{p}}}}$ such that $H^0 D_{\widehat{R_{\mathfrak{p}}}} \neq 0$ and $H^i D_{\widehat{R_{\mathfrak{p}}}} = 0$ for $i < 0$. The following is a dual result to the above theorem.

Theorem 2. *Let* $\mathsf{f} : \operatorname{Spec} R \to \mathbb{Z}$ *be a strictly increasing function. Then*

$$\prod_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\operatorname{ht}(\mathfrak{p})-\mathsf{f}(\mathfrak{p})} D_{\widehat{R_{\mathfrak{p}}}},$$

is a cosilting object in $D(R)$.

We will also explain an important connection between these theorems and the classification results in [1] and [4]. Moreover, we discuss when the two objects above become a tilting object and a cotilting object. A necessity condition is that $\mathsf{f} : \operatorname{Spec} R \to \mathbb{Z}$ is a *codimension function*, i.e., $\mathsf{f}(\mathfrak{q}) - \mathsf{f}(\mathfrak{p}) = 1$ whenever $\mathfrak{p} \subsetneq \mathfrak{q}$ and $\mathfrak{p}$ is maximal under $\mathfrak{q}$. Existence of a codimension function implies that R is catenary, so not every commutative noetherian ring can admit a codimension function. Existence of a dualizing complex implies that R has a codimension function f, and then we can prove that $\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\mathsf{f}(\mathfrak{p})} \mathbf{R}\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}$ and $\prod_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\operatorname{ht}(\mathfrak{p})-\mathsf{f}(\mathfrak{p})} D_{\widehat{R_{\mathfrak{p}}}}$ are a tilting object and a cotilting object, respectively.

On the other hand, we can also prove that if R is a homomorphic image of a Cohen-Macaulay ring of finite Krull dimension (which may not admits a dualizing complex but can admit a codimension function f), then $\bigoplus_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\mathsf{f}(\mathfrak{p})} \mathbf{R}\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}$ and $\prod_{\mathfrak{p}\in \operatorname{Spec} R} \Sigma^{\operatorname{ht}(\mathfrak{p})-\mathsf{f}(\mathfrak{p})} D_{\widehat{R_{\mathfrak{p}}}}$ are tilting and cotilting, respectively. This motivates us to characterize such a ring by tilting theoretic way via Kawasaki's work [3]. Indeed, this is an attempt trying to find a similar result to his deep work [2], which characterizes a homomorphic image of a Gorenstein ring of finite Krull dimension by existence of a dualizing complex.

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Positive cluster complexes and τ-tilting simplicial complexes

Yasuaki Gyoda

In this talk, we will give a generalization of Gabriel's theorem by using cluster algebra theory. Gabriel's theorem is the following theorem on the classification of path algebras of finite representation type, proved by Gabriel in 1972.

Theorem 1 ([4]).

(1) *For a connected quiver Q, the path algebra KQ of Q is of finite representation type if and only if Q is a Dynkin quiver of A,D or E type.*

(2) *If Q is a Dynkin quiver of A,D or E type, then the number of (isomorphic classes of) indecomposable modules depends only on underlying graph of Q and does not depend on the orienttation of Q.*

We focus on (2). In 1973, Bernstein-Gelfand-Ponomarev[2] gave another proof of this property by comparing module categories of KQ and KQ', where Q' is obtained from Q by reversing all arrows entering (or exiting) one sink (or source) vertex of Q. We call this opelation of a quiver the *sink/source mutation* of Q.

In this talk, I will introduce the following theorem, which generalizes path algebras to cluster-tilted algebras and indecomposable module to τ-rigid module:

Theorem 2 ([5]). *Let Λ and Λ' be cluster-tilted algebras of finite representation type such that the corresponding quiver Q'_Λ is obtained from Q_Λ by a sink or source mutation. Then Λ and Λ' has the same number of (isomorphic classes of) basic τ-rigid modules with k direct summands for any $k \in \mathbb{Z}_{\geq 0}$.*

Theorem 2 is a generalization of Theorem 1 (2), since Theorem 1 (2) follows immediately by setting Λ to be a path algebra of Dynkin type and $k = 1$. The key to proving Theorem 2 is two simplicial complexes, the *positive cluster complex* and the *τ-tilting simplicial complex*. The positive cluster complex is a special subcomplex of a cluster complex, which is a simplicial complex in cluster algebra theory. On the other hand, the τ-tilting simplicial complex is a simplicial complex whose vertex set consists of indecomposable τ-rigid modules and the simplicial set consists of basic τ-tilting modules. By the works of Adachi-Iyama-Reiten[1], Fu-Keller[6], and Cerulli Irelli-Keller-Labardini Fragoso-Plamondon[3], it is known that these two simplicial complexes coincide. Theorem 2 is proved by using cluster algebra theory and this coincidence.

In this talk, I will first give an overview of cluster algebra theory, then explain the cluster algebra version of Theorem 2, and show that Theorem 2 can be obtained by the simplicial complex correspondence described above.

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τ-tilting finite triangular matrix algebras

Takahiro Honma and Takuma Aihara

Classifying modules and subcategories with some good property is a central theme in representation theory. For example, a finitely generated progenerator, tilting modules, indecomposable modules , torsion classes and so on. Interestingly, modules and subcategory are closely related. In fact, a (classical) tilting module yields the torsion class. Moreover we can obtain more torsion classes by extending to support τ-tilting modules. If an algebra is τ-tilting finite, then there is a one-to-one correspondence between torsion classes and isomorphism classes of support τ-tilting modules. Hence, our aim is to investigate when an algebra is τ-tilting finite.

We focus on the τ-tilting finiteness of triangular matrix algebras. Recall that M. Auslander and I. Reiten showed that the representation finiteness of triangular matrix algebras is equivalent to the Auslander algebra in the paper [2]. We generalize the result to the τ-tilting finiteness. Furthermore, we explore the silting discreteness of tensor algebras over an algebraically closed field. Note that tensor algebras contain triangular matrix algebras.

This is a joint work with Takuma Aihara [1].

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On almost N-projective modules and generalized N-projective modules

Yoshiharu Shibata
Isao Kikumasa
Yosuke Kuratomi

Throughout this talk, R is a ring with identity and modules are unitary right R-modules. A module M is called *lifting* if, for any submodule N of M, there exists a direct sum decomposition $M = X \oplus Y$ such that $X \subseteq N$ and $N \cap Y \ll Y$. A module M is called *almost N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, either there exists a homomorphism $h : M \to N$ such that $f = gh$, or there exist a nonzero direct summand N' of N and a homomorphism $h' : N' \to M$ such that $g|_{N'} = fh'$. This projectivity was introduced by Harada and Tozaki [2] in 1989. After that, Baba and Harada [1] proved that a module $M = \bigoplus_{i=1}^{n} H_i$ where H_i is an LE-lifting module is lifting if and only if H_i is almost H_j-projective for $i \neq j$. In 2004, Mohamed and Müller [4] introduced the following projectivity: a module M is called *generalized N-projective* for a module N if, for any module X, any homomorphism $f : M \to X$ and any epimorphism $g : N \to X$, there exist direct sum decompositions $M = M_1 \oplus M_2$ and $N = N_1 \oplus N_2$, a homomorphism $h_1 : M_1 \to N_1$ and an epimorphism $h_2 : N_2 \to M_2$ such that $f|_{M_1} = gh_1$ and $g|_{N_2} = fh_2$. After that, an equivalent condition was given in [3] for a finite direct sum of lifting modules to be lifting with the finite internal exchange property in terms of relative generalized projectivity. Although a generalized N-projective module is almost N-projective, the converse is not true.

In this talk, we characterize these projectivities by certain coinvariants of homomorphisms between projective covers of M and N over a right perfect ring, and consider a condition for an almost N-projective module to be generalized N-projective. Our main results are the following:

Theorem A. Let R be a right perfect ring and let M and N be modules with the projective covers (P, ν_M) and (Q, ν_N), respectively. Then

(1) the following two conditions are equivalent:
  (a) M is almost N-projective,
  (b) for any $\alpha \in \mathrm{Hom}_R(P, Q)$, either $\alpha(\ker \nu_M) \subseteq \ker \nu_N$, or there exist $P' \leq_\oplus P$ and $Q' \leq_\oplus Q$ such that $\alpha|_{P'} : P' \to Q'$ is an isomorphism, $(\alpha|_{P'})^{-1}(\ker \nu_N|_{Q'}) \subseteq \ker \nu_M|_{P'}$ and $0 \neq \nu_N(Q') \leq_\oplus N$.

(2) the following two conditions are equivalent:
  (a) M is generalized N-projective,
  (b) for any $\alpha \in \mathrm{Hom}_R(P, Q)$, there exist decompositions $P = P_1 \oplus P_2$ and $Q = Q_1 \oplus Q_2$ such that $\alpha(P_1) \subseteq Q_1$, $\alpha(\ker \nu_M|_{P_1}) \subseteq \ker \nu_N|_{Q_1}$, $\alpha|_{P_2} : P_2 \to Q_2$ is an isomorphism, $(\alpha|_{P_2})^{-1}(\ker \nu_N|_{Q_2}) \subseteq \ker \nu_M|_{P_2}$, $M = \nu_M(P_1) \oplus \nu_M(P_2)$ and $N = \nu_N(Q_1) \oplus \nu_N(Q_2)$.

Theorem B. Let R be a right perfect ring and let M and N_i be modules $(i = 1, 2, \ldots, n)$. If M is lifting and almost N_i-projective, and N_i is almost N_j-projective for any distinct $i,\ j \in \{1, 2, \ldots, n\}$, then M is almost $\oplus_{i=1}^{n} N_i$-projective.

Theorem C. Let R be a right perfect ring, let M be a quasi-discrete module and let N be a lifting module. Then M is almost N-projective if and only if M is generalized N-projective.

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Tate-Hochschild cohomology and eventual periodicity for Gorenstein algebras

Satoshi Usui

In 1986, Buchweitz [1] introduced the notion of the *singularity categories* of Noetherian rings and used it to provide a framework for *Tate cohomology* of Gorenstein rings, which is a generalization of Tate cohomology of finite groups. Motivated by these notions due to Buchweitz, Wang [2] defined the *Tate-Hochschild cohomology groups* of a Noetherian algebra Λ over a field k to be

$$\widehat{\mathrm{HH}}^{i}(\Lambda) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(\Lambda \otimes_k \Lambda^{\mathrm{op}})}(\Lambda, \Lambda[i]),$$

where i is an integer, and $\mathcal{D}_{\mathrm{sg}}(\Lambda \otimes_k \Lambda^{\mathrm{op}})$ denotes the singularity category of the enveloping algebra $\Lambda \otimes_k \Lambda^{\mathrm{op}}$ of the algebra Λ. It was proved by Wang [2] that the $\mathbb{Z}$-graded vector space

$$\widehat{\mathrm{HH}}^{\bullet}(\Lambda) := \bigoplus_{i \in \mathbb{Z}} \widehat{\mathrm{HH}}^{i}(\Lambda)$$

carries a structure of a graded commutative algebra. We call this graded algebra the *Tate-Hochschild cohomology ring* of the algebra Λ. Moreover, he also proved in [3] that Tate-Hochschild cohomology rings are invariants under singular equivalence of Morita type with level, which is a new equivalence introduced by himself. Notice that recently such equivalences have been intensively studied. Therefore, it seems to be important to investigate the Tate-Hochschild cohomology rings.

In the last decade, the above rings have been studied for finite dimensional algebras. Dotsenko, Glinas and Tamaroff [4] showed that, for monomial Gorenstein algebras, their Tate-Hochschild cohomology rings have invertible homogeneous elements. Moreover, Usui [5] showed that the same statement holds for periodic algebras. In both cases, such invertible elements were obtained from the fact that each of the algebras is an *eventually periodic algebra*, that is, an algebra whose minimal projective resolution as a bimodule becomes periodic from some step.

In this talk, we first observe that there exists an eventually periodic algebra that is not Gorenstein, while it is well-known that periodic algebras are all Gorenstein algebras. We then characterize the eventual periodicity of a Gorenstein algebra as the existence of an invertible homogeneous element in the Tate-Hochschild cohomology ring of the Gorenstein algebra. Moreover, we provide examples of eventually periodic Gorenstein algebras (that are not periodic), and we explain that we can compute the Tate-Hochschild cohomology rings for periodic algebras by using their Hochschild cohomology rings.

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Generalizations of the correspondence between quasi-hereditary algebras and directed bocses

Yuichiro Goto

Quasi-hereditary algebras were introduced by Cline-Parshall-Scott to study the highest weight categories in Lie theory [3]. On the other hand, bocs theory was introduced by Roiter to solve tame and wild dichotomy and Crawley-Boevey applied it to analyze the module categories over tame algebras [2]. Koenig, Külshammer and Ovsienko connected these theories by giving equivalences between the categories of modules over directed bocses and those of Δ-filtered modules over quasi-hereditary algebras [4]. Moreover Brzeziński, Koenig and Külshammer showed that exact Borel subalgebras of quasi-hereditary algebras corresponding to directed bocses are homological [1]. Their results are as follows.

Theorem 1 ([1] Theorem 3.13, [4] Theorem 1.1, Corollary 1.3)**.** *We have a bijection*

$$\{\textit{Morita equivalence classes of quasi-hereditary algebras}\}$$
$$\updownarrow$$
$$\{\textit{Equivalence classes of the module categories over directed bocses}\}.$$

Let a quasi-hereditary algebra A and a direct bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra.

As natural generalizations of quasi-heredity algebras, there are two classes of algebras; Δ-filtered algebras (or standardly stratified algebras) and $\overline{\Delta}$-filtered algebras. Our interest is to generalize the above theorem for those algebras. In the case where the algebra is Δ-filtered, we obtain the result below by using arguments similar to those of [4]. But for $\overline{\Delta}$-filtered algebras, there are two problems when we apply their arguments. In the talk, we explain how to avoid them. Here is the main result.

Theorem 2. *We have a bijection*

$$\{\textit{Morita equivalence classes of } \Delta\textit{-filtered (resp. } \overline{\Delta}\textit{-filtered) algebras}\}$$
$$\updownarrow$$
$$\{\textit{Equivalence classes of the module categories over weakly directed (resp. one-cyclic directed) bocses}\}.$$

Let a Δ-filtered (resp. $\overline{\Delta}$-filtered) algebra A and a weakly direct (resp. one-cyclic directed) bocs $\mathcal{B}$ correspond via the above bijection. Then the right Burt-Butler algebra $R_{\mathcal{B}}$ of $\mathcal{B}$ is Morita equivalent to A. Moreover, $R_{\mathcal{B}}$ has a homological exact Borel subalgebra (resp. a homological proper Borel subalgebra).

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NUMERICAL TORSION PAIRS AND CANONICAL DECOMPOSITIONS FOR ELEMENTS IN THE GROTHENDIECK GROUP

Sota Asai, Osamu Iyama

Let A be a finite-dimensional algebra over an algebraically closed field K. Then, the Grothendieck group $K_0(\mathsf{proj}\,A)$ of the category $\mathsf{proj}\,A$ of finitely generated projective A-modules is a free abelian group, whose canonical $\mathbb{Z}$-basis is given by the non-isomorphic indecomposable projective modules $P_1, P_2, \dots, P_n$. Let S_i be the simple top of P_i for each i, then $S_1, S_2, \dots, S_n$ are the non-isomorphic simple modules, and they give the canonical $\mathbb{Z}$-basis of the Grothendieck group $K_0(\mathsf{mod}\,A)$ of the category $\mathsf{mod}\,A$ of finitely generated A-modules.

The *Euler form* is the $\mathbb{Z}$-bilinear form $\langle ?, ! \rangle \colon K_0(\mathsf{proj}\,A) \times K_0(\mathsf{mod}\,A) \to \mathbb{Z}$ satisfying $\langle P_i, S_j \rangle = \delta_{i,j}$. Via the Euler form, every element $\theta \in K_0(\mathsf{proj}\,A)$ can be identified with the $\mathbb{Z}$-linear form $K_0(\mathsf{mod}\,A) \to \mathbb{Z}$. These are obviously extended to the real Grothendieck groups $K_0(\mathsf{proj}\,A)_{\mathbb{R}} := K_0(\mathsf{proj}\,A) \otimes_{\mathbb{Z}} \mathbb{R}$ and $K_0(\mathsf{mod}\,A)_{\mathbb{R}} := K_0(\mathsf{mod}\,A) \otimes_{\mathbb{Z}} \mathbb{R}$. We use these Grothendieck groups to study the *torsion pairs* in $\mathsf{mod}\,A$.

To each element $\theta \in K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, we can associate the *$\theta$-semistable subcategory* $\mathcal{W}_\theta$ by King [5] and the *numerical torsion pairs* $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ by Baumann–Kamnitzer–Tingley [2]. These categories are defined by linear inequalities given by θ on the dimension vectors of submodules and factor modules of each module. By using θ-semistable subcategories, Brüstle–Smith–Treffinger [3] considered a *wall-chamber structure* on $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, which is identified with the Euclidean space $\mathbb{R}^n$. In [1], the first named speaker studied this wall-chamber structure by considering the equivalence relation called *TF equivalence* on $K_0(\mathsf{proj}\,A)_{\mathbb{R}}$, where θ and η are said to be *TF equivalent* if $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_\eta, \mathcal{F}_\eta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_\eta, \overline{\mathcal{F}}_\eta)$.

On the other hand, for each $\theta \in K_0(\mathsf{proj}\,A)$, we can take $P_1, P_0 \in \mathsf{proj}\,A$ such that $\theta = [P_0] - [P_1]$ and that P_0 and P_1 have no nonzero common direct summand. We define the *presentation space* $\mathsf{Hom}(\theta) := \mathsf{Hom}_A(P_1, P_0)$. Then, each $f \in \mathsf{Hom}(\theta)$ gives two torsion pairs $(\overline{\mathcal{T}}_f, \mathcal{F}_f)$ and $(\mathcal{T}_f, \overline{\mathcal{F}}_f)$ via $\mathsf{Coker}\,f$ and $\mathsf{Ker}\,\nu f$. We compare them to numerical torsion pairs in our study.

Moreover, every $f \in \mathsf{Hom}(\theta)$ defines a 2-term complex $P_f := (P_1 \xrightarrow{f} P_0)$ in the homotopy category $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$. Based on how the complex P_f is decomposed into indecomposable direct summands in $\mathsf{K}^{\mathrm{b}}(\mathsf{proj}\,A)$, Derksen–Fei [4] introduced the notions of *indecomposable* elements and the *canonical decompositions* $\theta = \bigoplus_{i=1}^m \theta_i$ in the Grothendieck group $K_0(\mathsf{proj}\,A)$. Derksen–Fei and Plamondon [6] showed that any element $\theta \in K_0(\mathsf{proj}\,A)$ admits a unique canonical decomposition up to reordering. We investigate also the relationship between canonical decompositions and numerical torsion pairs.

We will talk about some of our important results on these topics.

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Subcategories and silting objects of Noetherian algebras

Osamu Iyama and Yuta Kimura

A *torsion class* (resp. *torsionfree class*) is a full subcategory of an abelian category $\mathcal{A}$ which is closed under extensions and factor objects (resp. subobjects). It appears in several branches of mathematics and plays important roles.

When $\mathcal{A} = \mathsf{mod}A$ for a finite dimensional algebra A over a field, there are many studies of subcategories of $\mathcal{A}$. Among others, a connection between torsion classes and classical tilting modules was well understood in the last century. τ-tilting modules [1] (which we also call *silting modules*) plays a central role in the recent development of tilting theory from a point of view of mutation.

When $\mathcal{A} = \mathsf{mod}R$ for a commutative Noetherian ring R, classification problems of subcategories of $\mathcal{A}$ have been studied in many mathematicians. Serre subcategories and torsion classes of $\mathcal{A}$ bijectively correspond to specialization closed subsets of $\mathsf{Spec}R$, by Gabriel [2], Stanley-Wang [5]. Moreover, torsionfree classes of $\mathsf{mod}R$ bijectively corresponds to subsets of $\mathsf{Spec}R$ by Takahashi [6].

In this talk, we study and develop silting theory and classification problems of subcategories for Noetherian algebras. Let R be a commutative Noetherian ring. A *Noetherian R-algebra* is an R-algebra which is finitely generated as an R-module. For a Noetherian R-algebra Λ, we study and classify subcategories of $\mathsf{mod}\Lambda$ by comparing them with subcategories of modules over finite dimensional algebras. We denote by $\mathsf{tors}\Lambda$ (resp. $\mathsf{torf}\Lambda$) the set of torsion (resp. torsionfree) classes of $\mathsf{mod}\Lambda$. Let $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ and $\mathbb{T}_R(\Lambda) = \prod_{\mathfrak{p}} \mathsf{tors}(\Lambda \otimes_R \kappa(\mathfrak{p}))$, $\mathbb{F}_R(\Lambda) = \prod_{\mathfrak{p}} \mathsf{torf}(\Lambda \otimes_R \kappa(\mathfrak{p}))$, where $\mathfrak{p}$ runs all prime ideals of R. Note that these sets are partially ordered by inclusion.

Theorem 1. *For a Noetherian R-algebra Λ, the following statements hold.*

(1) *There is an isomorphism of posets* $\mathsf{torf}\Lambda \xrightarrow{\sim} \mathbb{F}_R(\Lambda)$.

(2) *There is an injective morphism of posets* $\Phi : \mathsf{tors}\Lambda \longrightarrow \mathbb{T}_R(\Lambda)$.

As a corollary, we classify Serre subcategories of $\mathsf{mod}\Lambda$ by using simple $(\Lambda \otimes_R R_{\mathfrak{p}})$-modules ($\mathfrak{p} \in \mathsf{Spec}R$). These results recover bijections by Takahashi, Stanley-Wang and Gabriel.

We next study the image of Φ. We introduce the notion of compatible elements in $\mathbb{T}_R(\Lambda)$, and prove that all elements in the image are compatible. We give a sufficient condition on Λ such that all compatible elements belong to the image. To do this, we characterize the finiteness of $\mathsf{tors}(\Lambda \otimes_R \kappa(\mathfrak{p}))$ instead of silting modules over $\Lambda \otimes_R R_{\mathfrak{p}}$. In particular, we have the following formula.

Theorem 2. *Let R be a commutative Noetherian ring and Q be a Dynkin quiver. Then there is an isomorphism of posets*

$$\mathsf{tors}(RQ) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{poset}}(\mathsf{Spec}R, \mathfrak{C}_Q),$$

where $\mathfrak{C}_Q$ is the Cambrian lattice of Q.

This talk is based on a preprint [3].

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Structure theorem for flat cotorsion modules over Noether algebras

Ryo Kanda and Tsutomu Nakamura

A right module M over a ring A is called *cotorsion* if $\mathrm{Ext}^1_A(F, M) = 0$ for all flat right A-modules F. A *flat cotorsion* module is a module being flat and cotorsion. The *flat cover conjecture*, which was affirmatively solved by Bican, El Bashir, and Enochs [BEBE01], implies that the class of flat modules and the class of cotorsion modules form a complete cotorsion pair. Flat cotorsion modules are those modules that belong to the core of this cotorsion pair.

Enochs [Eno84] gave a structure theorem for flat cotorsion modules over a commutative noetherian ring R: An R-module M is flat cotorsion if and only if M is isomorphic to

$$\prod_{\mathfrak{p}\in \mathrm{Spec}\, R} \mathrm{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{p})^{(B_\mathfrak{p})})$$

for some family of sets $\{B_\mathfrak{p}\}_{\mathfrak{p}\in \mathrm{Spec}\, R}$, where $E_R(R/\mathfrak{p})$ is the injective envelope of $R/\mathfrak{p}$ and $E_R(R/\mathfrak{p})^{(B_\mathfrak{p})}$ is the direct sum of its B_P-indexed copies. The cardinality of each $B_\mathfrak{p}$ is uniquely determined by the isomorphism class of M.

We establish a noncommutative generalization of Enochs' structure theorem. For a commutative noetherian ring R, a *Noether R-algebra* is a ring A together with a ring homomorphism $\varphi\colon R \to A$ such that the image of φ is contained in the center of A and A is finitely generated as an R-module. $\mathrm{Spec}\, A$ denotes the set of prime (two-sided) ideals of A.

For each $P \in \mathrm{Spec}\, A$, denote by $I_A(P)$ the corresponding indecomposable injective right A-module, whose only associated prime is P. Note that $P \cap R := \varphi^{-1}(P)$ is a prime ideal of R.

Theorem 1. *Let A be a Noether R-algebra. A right A-module M is flat cotorsion if and only if M is isomorphic to*

$$\prod_{P\in \mathrm{Spec}\, A} \mathrm{Hom}_R(I_{A^{\mathrm{op}}}(P), E_R(R/(R\cap P))^{(B_P)})$$

for some family of sets $\{B_P\}_{P\in \mathrm{Spec}\, A}$. The cardinality of each B_P is uniquely determined by the isomorphism class of M.

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On τ-tilting finiteness of tensor product algebras between simply connected algebras

Kengo Miyamoto and Qi Wang

This talk is based on joint work with Qi Wang ([1]).

Throughout, we will use the symbol k to denote an algebraically closed field, and tensor products are always taken over k. An algebra is always assumed to be an associative basic ring-connected simply connected finite-dimensional k-algebra.

An algebra A is *τ-tilting finite* if it has only finitely many support τ-tilting modules. In this talk, we discuss the τ-tilting finiteness for the tensor product $A \otimes B$ between two τ-tilting finite simply connected algebras A and B. If A is a simply connected path algebra, we have the following result.

Theorem 1. *Let A be a path algebra of finite connected acyclic quiver with $n \geq 2$ simple modules. Then, the following statements hold.*
(1) If B is hereditary, then $A \otimes B$ is τ-tilting finite if and only if $A \simeq \mathsf{k}(1 \to 2)$ and B is isomorphic to one of path algebras of $\mathbb{A}_2$, $\mathbb{A}_3$ or $\mathbb{A}_4$.
(2) Let B be a simply connected algebra. If $\mathsf{k}(1 \to 2) \otimes B$ is τ-tilting finite, then any connected component of the separated quiver of the quiver of B is of type $\mathbb{A}_n$.
(3) Assume that $n \geq 3$ and B is a simply connected algebra which is not hereditary. Then, $A \otimes B$ is τ-tilting finite if and only if A is isomorphic to a path algebra of $\mathbb{A}_3$ and B is isomorphic to a Nakayama algebra with radical square zero.

In the case that both A and B are not hereditary, we may give a visualization table below to illustrate the τ-tilting finiteness of $A \otimes B$. In the table below, F means τ-tilting finite, IF means τ-tilting infinite, and "F or IF" means that there are both cases. We denote by $\mathsf{rad}(A)$ the Jacobson radical of A and by $\lvert A\rvert$ the number of isomorphism classes of simple A-modules.

$A \otimes B$ (A, B: simply connected)			B :Nakayama			B:Not Nakayama		
			$\mathsf{rad}^2 = 0$		$\mathsf{rad}^2 \neq 0$			
			$n = 3$	$n \geq 4$		$\lvert B\rvert = 3$	$\lvert B\rvert = 4$	$\lvert B\rvert \geq 5$
A:Nakayama	$\mathsf{rad}^2 = 0$	$n = 3$	F	F	Open	F	F or IF	F or IF
		$n \geq 4$	F	F	F or IF	F	F or IF	IF
	$\mathsf{rad}^2 \neq 0$		Open	F or IF	IF	IF	IF	IF
A:Not Nakayama		$\lvert A\rvert = 3$	F	F	IF	IF	IF	IF
		$\lvert A\rvert = 4$	F or IF	F or IF	IF	IF	IF	IF
		$\lvert A\rvert \geq 5$	F or IF	IF	IF	IF	IF	IF

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Algebras associated to Noncommutative Conics in Quantum Projective Planes

Haigang Hu

Let k be an algebraically closed field of characteristic 0. Algebras are all over k. A connected graded algebra A is a positively graded algebra $A = \oplus_{n \geq 0} A_n$ such that $A_0 = k$.

In (commutative) algebraic geometry, it is important to study the homogeneous coordinate ring $k[x_1, \cdots, x_n]/(f)$ of a quadric hypersurface in the projective space $\mathbb{P}^{n-1}$ where $0 \neq f \in k[x_1, \cdots, x_n]_2$. In noncommutative algebraic geomtry, we say the quotient algebra $S/(f)$ the *noncommutative quadric hypersurface* (*noncommutative conic* if $d = 3$) where S is a d-dimensional quantum polynomial algebra defined below and $0 \neq f \in S_2$ a regular central element.

Definition 1. A noetherian connected graded algebra S generated in degree 1 is called a *d-dimensional quantum polynomial algebra* if (1) $\operatorname{gldim} S = d$, (2) $\operatorname{Ext}^i_S(k, S(-j)) \cong \begin{cases} k & \text{if } i = j = d, \\ 0 & \text{otherwise,} \end{cases}$ and (3) $H_S(t) := \sum_{i=0}^{\infty} (\dim_k S_i) t^i = 1/(1-t)^d$.

The classification of noncommutative quadric hypersurfaces is a big project in noncommutative algebraic geometry and it is far away from complete. The good thing is there are many notable developments in the study of noncommutative quadric hypersurfaces: Smith and Van den Bergh introduce a finite dimensional algebra $C(A)$ associated to $A := S/(f)$ which determines the Cohen-Macaulay representation of A (cf. [5]); Mori and Ueyama introduce the noncommutative matrix factorization of f over S and they also proved the noncommutative Knörrer's periodicity theorem (cf. [4]). He and Ye introduce the Clifford deformation associated to the pair (S, f) which is a nonhomogeneous PBW deformation and they showed that A is a noncommutative isolated singularity if and only if $C(A)$ is semisimple (cf. [1]), etc.

However, there is still no complete classification of noncommutative conics even though it should be the easiest case. Using theoretical tools above, and the fact that defining relations of all 3-dimensional quantum polynomial algebras are given by Itaba and Matsuno (cf. [3]), it is time for us to begin to work on the classification of noncommutative conics and it would be a good step forward to classify noncommutative quadric hypersurfaces. In [2], we focus on the study of a noncommutative conic A such that its quadratic dual $A^!$ is commutative. The following is the main result in this talk.

Theorem 2 ([2])**.** *Let $A = S/(f)$ be a noncommutative conic. Then $A^!$ is commutative if and only if S is determined by a symmetric regular superpotential. Moreover, the set of isomorphism classes of associated algebras $C(A)$ is equal to the set of isomorphism classes of 4-dimensional commutative Frobenius algebras. They are:*

$$k^4, k[u]/(u^2) \times k^2, (k[u]/(u^2))^{\times 2}, k[u]/(u^3) \times k, k[u]/(u^4), k[u,v]/(u^2, v^2).$$

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Characterization of the quantum projective planes finite over their centers

Ayako Itaba and Izuru Mori

This talk is based on [5]. A quantum polynomial algebra A is a noncommutative analogue of a commutative polynomial algebra, and a quantum projective space $\mathsf{Proj}_{\mathrm{nc}} A$ in the sense of Artin-Zhang [2] is the noncommutative projective scheme associated to a quantum polynomial algebra A. So, they are the most basic objects to study in noncommutative algebraic geometry.

For a 3-dimensional quantum polynomial algebra $A = \mathcal{A}(E, \sigma)$, Artin-Tate-Van den Bergh [3] showed that A is finite over its center if and only if the order $|\sigma|$ of σ is finite. Also, Artin [1] showed that if A is finite over its center and E is not a projective plane $\mathbb{P}^2$, then A has a fat point module, which plays an important role in noncommutative algebraic geometry, however, the converse is not true in general. To check the existence of a fat point, the following notion is defined in [6];

$$\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i = \phi|_E \text{ for some } \phi \in \mathrm{Aut}\mathbb{P}^2\}.$$

In fact, $\mathsf{Proj}_{\mathrm{nc}} A$ has a fat point if and only if $1 < \|\sigma\| < \infty$ ([1], [6]). Moreover, in [6], the notion that $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center was introduced, and the following result was proved; let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra such that $E \subset \mathbb{P}^2$ is a triangle. Then $\|\sigma\| < \infty$ if and only if $\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center. The purpose of this research is to extend the above theorem to all 3-dimensional quantum polynomial algebras, and the following is our main result.

Theorem 1. *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra such that $E \neq \mathbb{P}^2$, and $\nu \in \mathrm{Aut}\, A$ the Nakayama automorphism of A. Then the following are equivalent:*

(1) $|\nu^* \sigma^3| < \infty$.
(2) $\|\sigma\| < \infty$.
(3) $\mathsf{Proj}_{\mathrm{nc}} A$ *is finite over its center.*
(4) $\mathsf{Proj}_{\mathrm{nc}} A$ *has a fat piont.*

As a byproduct, we prove that $|\nu^* \sigma^3|$ is 1 or infinite if and only if the isomorphism classes of simple 2-regular modules over the Beilinson algebra ∇A of A are parameterized by $E \subset \mathbb{P}^2$. Note that ∇A is a typical example of 2-representation infinite algebra defined in Herschend-Iyama-Oppermann [4].

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Localization of extriangulated categories

Arashi Sakai

Abelian categories, exact categories and triangulated categories are the main categorical frameworks used in homological algebra. Localization of these categories are frequently used in representation theory, for example, the Serre quotient of abelian categories [1] and the Verdier quotient of triangulated categories [6]. In recent years, the notion of extriangulated categories is introduced in [4] and unifies exact categories and triangulated categories. So far we have not seen localization of extriangulated categories in the literature.

In this talk, we introduce localization of extriangulated categories. In particular, we consider localization arising from biresolving subcategories [5] and percolating subcategories [2].

Theorem 1. *Let $\mathcal{C}$ be an extriangulated category and $\mathcal{N}$ a biresolving or percolating thick subcategory. Then there is a class of morphism $\mathcal{S}_{\mathcal{N}}$ in $\mathcal{C}$, and localization $\mathcal{C}_{\mathcal{S}_{\mathcal{N}}}$ of $\mathcal{C}$ by $\mathcal{S}_{\mathcal{N}}$ has a natural extriangulated structure.*

This localization covers the Serre quotient, the Verdier quotient and several other localizations in the special cases and has the universality in some sense. This talk is based on [3], joint work with Hiroyuki Nakaoka (Nagoya University) and Yasuaki Ogawa (Nara University of Education).

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Some classes of subcategories of module categories: classifications and the relation between them

Haruhisa Enomoto

In the representation theory of algebras, the study of subcategories of module categories have been one of the main topics. Among them, *torsion classes* (subcategories closed under quotients and extensions) have been attracted an attention. There are another class of subcategories of module categories: *wide subcategories* (subcategories closed under kernels, cokernels, and extensions). They are related to ring epimorphisms of an algebra, and the relation between torsion classes and wide subcategories have been studied by several authors.

Recently, I introduced some classes of subcategories of module categories which generalizes both torsion classes (or torsion-free classes) and wide subcategories: ICE-closed subcategories (Image-Cokernel-Extension-closed), and its dual IKE-closed subcategories (Image-Kernel-Extension-closed), hearts of intervals of torsion classes. They are particular classes of extension-closed subcategories which can be *controlled by torsion classes*. In this talk, I will talk about the classification results of these subcategories, and discuss the relation between these classes of subcategories, based on [1, 2, 3]. In particular, we can recover the poset structure of these subcategories using only the poset structure of torsion classes.

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Intervals of s-torsion pairs in extriangulated categories with negative first extensions

Takahide Adachi, Haruhisa Enomoto and Mayu Tsukamoto

Happel, Reiten and Smalø [2] provided a construction of new t-structures through torsion pairs in the heart of a given t-structure. This construction induces a close connection between t-structures and torsion pairs as follows.

Theorem 1 ([2, 4])**.** *Let $\mathcal{D}$ be a triangulated category with shift functor Σ. Let $(\mathcal{U}, \mathcal{V})$ be a t-structure on $\mathcal{D}$ and $\mathcal{H} := \mathcal{U} \cap \Sigma\mathcal{V}$ the heart of $(\mathcal{U}, \mathcal{V})$. Then there exists a poset isomorphism between the poset of t-structures $(\mathcal{U}', \mathcal{V}')$ on $\mathcal{D}$ satisfying $\Sigma\mathcal{U} \subseteq \mathcal{U}' \subseteq \mathcal{U}$ and the poset of torsion pairs in $\mathcal{H}$.*

Let $t_1 := (\mathcal{T}_1, \mathcal{F}_1)$ and $t_2 := (\mathcal{T}_2, \mathcal{F}_2)$ be torsion pairs in an abelian category $\mathcal{A}$ with $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Let $[t_1, t_2]$ denote the *interval* in the poset of torsion pairs in $\mathcal{A}$ consisting of $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T}_1 \subseteq \mathcal{T} \subseteq \mathcal{T}_2$. We call the subcategory $\mathcal{H}_{[t_1,t_2]} := \mathcal{T}_2 \cap \mathcal{F}_1$ the *heart* of $[t_1, t_2]$. The following bijection induces fruitful results for the poset structure of torsion pairs in $\mathcal{A}$.

Theorem 2 ([1, 3])**.** *Let $\mathcal{A}$ be an abelian category and $[t_1, t_2]$ an interval in the poset of torsion pairs in $\mathcal{A}$. Then there exists a poset isomorphism between $[t_1, t_2]$ and the poset of torsion pairs in $\mathcal{H}_{[t_1,t_2]}$.*

The aim of this talk is to show that two poset isomorphisms in Theorem 1 and Theorem 2 are consequences of a more general poset isomorphism in extriangulated categories. Since t-structures are exactly torsion pairs whose negative first extensions vanish, we introduce *negative first extensions in extriangulated categories*, that is, an additive bifunctor $\mathbb{E}^{-1}$ satisfying a certain condition. We can naturally regard triangulated categories and exact categories as extriangulated categories with negative first extensions. As a common generalization of t-structures and torsion pairs (in abelian categories), we introduce the notion of s-torsion pairs. We call a pair $(\mathcal{T}, \mathcal{F})$ of subcategories an *s-torsion pair* if it is a torsion pair (in the usual sense) and $\mathbb{E}^{-1}(\mathcal{T}, \mathcal{F}) = 0$ holds. Since the set of s-torsion pairs becomes a partially ordered set by inclusions, we can define the *hearts* of intervals as with torsion pairs in abelian categories. Moreover, each heart can be naturally regarded as an extriangulated category with a negative first extension. In this setting, we obtain the following result.

Theorem 3. *Let $\mathcal{C}$ be an extriangulated category with a negative first extension. Let $[t_1, t_2]$ be an interval in the poset of s-torsion pairs in $\mathcal{C}$ and $\mathcal{H}_{[t_1,t_2]}$ its heart. Then there exists a poset isomorphism between $[t_1, t_2]$ and the poset of s-torsion pairs in $\mathcal{H}_{[t_1,t_2]}$.*

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Examples of tilting-discrete self-injective algebras

Takahide Adachi and Ryoichi Kase

In the representation theory of algebras, the notion of mutation, which is an operation to construct a new object from an original one by exchanging direct summands, plays a crucial role. From the viewpoint of mutation, there are two important classes: one is silting-discrete algebras and the other is tilting-discrete self-injective algebras. A finite dimensional algebra is called a *silting-discrete* (respectively, *tilting-discrete*) algebra if for each positive integer d, the set of isomorphism classes of basic d-term silting (respectively, tilting) objects of the bounded homotopy category of finitely generated projective modules is finite. As a nice property of silting-discrete (respectively, tilting-discrete self-injective) algebras, any two silting (respectively, tilting) objects are obtained from each other by iterated silting (respectively, tilting) mutation ([1, 2, 3]). By definition, silting-discrete algebras always are tilting-discrete. However, we do not know whether tilting-discrete algebras are silting-discrete. Now we propose a natural question: Is a tilting-discrete algebra always silting-discrete?

Our aim of this talk is to give two examples for the question.

Theorem 1. *Let A be a basic connected non-local self-injective algebra over an algebraically closed field and let $\{e_1, e_2, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents of A. Assume that A admits a cyclic Nakayama permutation and* $\mathrm{soc}(e_iA) \subset \mathrm{rad}^2 A$ *for all i. Then there exists a self-injective algebra $\widetilde{A}$ such that*

- *it is not silting-discrete,*
- *$\{\widetilde{A}[i] \mid i \in \mathbb{Z}\}$ coincides with the set of isomorphism classes of all basic tilting objects for $\widetilde{A}$. In particular, $\widetilde{A}$ it tilting-discrete.*

Theorem 2. *Let K be an algebraically closed field and let $n, m \geq 5$ be integers with* $\gcd(n-1, m) = 1$. *Assume that n is odd and m is not divisible by the characteristic of K. Let $A_{n,m}$ be the stable Auslander algebra of a self-injective Nakayama K-algebra with m simple modules (up to isomorphism) and Loewy length n. Then $A_{n,m}$ is a tilting-discrete self-injective algebra but not silting-discrete.*

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The structure of Adams graded dg algebras and Cohen-Macaulay representations

Norihiro Hanihara

The subject of this talk is Cohen-Macaulay representation theory of Gorenstein rings, from the viewpoint of tilting and cluster tilting theory. While the derived category $\mathrm{D^b}(\mathrm{mod}\,A)$ of an algebra A is a triangulated category of the canonical form endowed with a tilting object, the d-cluster category $\mathrm{C}_d(A)$ of A can be seen as a counterpart for d-Calabi-Yau triangulated categories with d-cluster tilting objects.

One of the (big) problems on this subject (see [2]) is to find triangle equivalences between the stable category of graded Cohen-Macaulay modules and the derived category of an algebra, and between the ungraded stable category and the cluster category, realizing the stable categories as "canonical forms" of triangulated categories.

The ideal situation is summarized in the following commutative diagram, where R is a nice graded Gorenstein ring, and A is a finite dimensional algebra.

$$\begin{array}{ccc} \underline{\mathrm{CM}}^{\mathbb{Z}}R & \overset{\simeq}{\text{———}} & \mathrm{D^b}(\mathrm{mod}\,A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}\,R & \overset{\simeq}{\text{———}} & \mathrm{C}_d(A) \end{array}$$

Such equivalences provide a mutual understanding of representation theories of R and A, and can also explain certain classification results (e.g. [3, 4]).

While the "graded–derived" equivalence naturally predicts the "ungraded–cluster" equivalence, it often involves technical difficulties to actually prove. The following consequence of our result says that the latter equivalence follows automatically from the former one.

Theorem 1. *Let $R = \bigoplus_{i\geq 0} R_i$ be a positively graded commutative Gorenstein isolated singularity of dimension $d+1$, and of Gorenstein parameter $a \neq 0$ such that each R_i is finite dimensional over a field. Suppose that $\underline{\mathrm{CM}}^{\mathbb{Z}}R$ has a tilting object M such that the endomorphism algebra $A = \underline{\mathrm{End}}_R^{\mathbb{Z}}(M)$ has global dimension at most d. Then there exists a commutative diagram of equivalences*

$$\begin{array}{ccc} \underline{\mathrm{CM}}^{\mathbb{Z}}R & \overset{\simeq}{\text{———}} & \mathrm{D^b}(\mathrm{mod}\,A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}^{\mathbb{Z}/a\mathbb{Z}}R & \overset{\simeq}{\text{———}} & \mathrm{C}_d(A) \\ \downarrow & & \downarrow \\ \underline{\mathrm{CM}}\,R & \overset{\simeq}{\text{———}} & \mathrm{C}_d^{(1/a)}(A). \end{array}$$

Here, the category $\mathrm{C}_d^{(1/a)}(A)$ is the triangulated hull of the orbit category of $\mathrm{D^b}(\mathrm{mod}\,A)$ by a naturally defined a-th root of $\nu_d = - \otimes_A^{\mathrm{L}} DA[-d]$ (c.f. [1]). The key ingredient toward the above theorem is differential graded (=dg) enhancements of singularity categories of commutative Gorenstein rings (or more generally symmetric orders), and certain Calabi-Yau property of these dg categories. We obtain the above result as a consequence of a structure theorem for Adams graded Calabi-Yau dg categories.

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Tilting Theory for Periodic Triangulated Categories

Shunya Saito

Tilting theory gives a way to relate a triangulated category with the derived category of an algebra, and play a central role in representation theory of algebras. However, it does not work *m-periodic triangulated category*. A triangulated category $\mathcal{T}$ is m-periodic if its suspension functor Σ satisfies $\Sigma^m \simeq 1_{\mathcal{T}}$ as additive functors. Such categories naturally arise in representation theory of self-injective algebras and hypersurface singularities.

In this talk, we will talk about a periodic analogue of tilting theory. We will explain *m-periodic tilting theorem*, which asserts that an m-periodic triangulated category having an *m-periodic tilting object* is equivalent to the *m-periodic derived category* under some assumptions. As an application, we give an equivalence between the stable module category of a self-injective algebra and the 2-periodic derived category of a hereditary algebra. This talk based on [1].

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Quaternion rings over local rings

Isao Kikumasa and Kiyoichi Oshiro

In 1843, Hamilton discovered the 4-dimensional division algebra $H(\mathbb{R}) = \mathbb{R} \oplus i\mathbb{R} \oplus j\mathbb{R} \oplus k\mathbb{R}$ over the field $\mathbb{R}$ of real numbers. Hamilton's great discovery is the following beautiful multiplications for the basis $\{1, i, j, k\}$:

$$i^2 = j^2 = k^2 = ijk = -1 \quad \cdots\cdots \ (*).$$

Starting from given any ring R and a free right R-module $R \oplus iR \oplus jR \oplus kR$, the quaternion ring $H(R)$ is canonically defined by the multiplications $(*)$. For nonzero elements a, b in the center of R, the generalized quaternion ring $H(R; a, b)$ is defined. In particular, $H(R; -1, -1)$ is $H(R)$. For a commutative field F with $2 \neq 0$, $H(F)$ is a division ring or isomorphic to the ring of 2×2 matrices over F, that is a well known classical theorem. In [1], [2] and [4], quaternion rings and generalized quaternion rings over division rings or other rings are studied.

This talk is based on [3]. In this talk, from ring theoretic view points, we give fundamental results on $H(R)$ and $H(R; a, b)$ over a local ring R, from which we can look over several known results over fields or division rings.

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On support τ-tilting modules and semibricks for blocks of group algebras

Ryotaro Koshio and Yuta Kozakai

Let k be an algebraically closed field of characteristic $p > 0$, $\tilde{G}$ a finite group and G a normal subgroup of $\tilde{G}$. Moreover let B be a block of kG and $\tilde{B}$ a block of $k\tilde{G}$ covering B, that is, a block of kG satisfying the condition that $1_B 1_{\tilde{B}} \neq 0$, where 1_B and $1_{\tilde{B}}$ are units of B and $\tilde{B}$ respectively.

The notion of support τ-tilting modules was introduced in [1] and has been studied by many researchers. These modules play important roles in representation theory of finite dimensional algebras because they correspond to many classes of representation theoretical objects, for example the two-term silting complexes, torsion classes [1], semibricks [2] and more.

In this talk, we focus on the support τ-tilting modules and semibricks over the block $\tilde{B}$ under the assumption that the quotient group $\tilde{G}/G$ is a p-group or a cyclic group of a p-prime order, and we compare the support τ-tilting modules and semibricks over the block $\tilde{B}$ to those over the block B, which is easier to consider than $\tilde{B}$. Moreover, based on [3], in the case of the quotient group $\tilde{G}/G$ being a p-group, we explain that the set of support τ-tilting modules over $\tilde{B}$ is isomorphic to that over B as partially ordered sets and that all semibricks over $\tilde{B}$ can be obtained from the extensions of those over B.

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Hochschild cohomology of N_m

Tomohiro Itagaki, Kazunori Nakamoto, and Takeshi Torii

Let R be a commutative ring. Let $m \geq 3$. Set

$$\mathrm{N}_m(R) = \{(a_{ij}) \in \mathrm{M}_m(R) \mid a_{11} = a_{22} = \cdots = a_{mm} \text{ and } a_{ij} = 0 \text{ for any } i > j\}.$$

Setting $x_1 = E_{1,2}, x_2 = E_{2,3}, \ldots, x_{m-1} = E_{m-1,m}$, we have an isomorphism as R-algebras:

$$\mathrm{N}_m(R) \cong R\langle x_1, x_2, \ldots, x_{m-1}\rangle / \langle x_i x_j \mid j \neq i+1\rangle.$$

The Koszul dual $\mathrm{N}_m(R)^!$ of $\mathrm{N}_m(R)$ is isomorphic to $R\langle y_1, y_2, \ldots, y_{m-1}\rangle/\langle y_i y_{i+1} \mid 1 \leq i \leq m-2\rangle$. Put

$$\varphi(d) = \mathrm{rank}_R \mathrm{N}_m(R)^!_d,$$

where $|y_i| = 1$ and $\mathrm{N}_m(R)^!_d$ is the homogeneous part of $\mathrm{N}_m(R)^!$ of degree d. The Poincaré series $f(t) = \sum_{d \geq 0} \varphi(d) t^d$ can be calculated by $f(t) = 1/(1 + \sum_{k=1}^{m-1} (-1)^k (m-k) t^k)$.

Theorem 1. *The Hochschild cohomology* $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ *is a free R-module for $n \geq 0$. The rank of* $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R))$ *for $n \geq 0$ is given by*

$$\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{M}_m(R)/\mathrm{N}_m(R)) = \begin{cases} m-1 & (n=0) \\ (m-2)\varphi(n) & (n>0). \end{cases}$$

Theorem 2. *The Hochschild cohomology* $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ *is a free R-module for $n \geq 0$. The rank of* $\mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R))$ *is given by*

$$\begin{aligned} &\mathrm{rank}_R \mathrm{HH}^n(\mathrm{N}_m(R), \mathrm{N}_m(R)) \\ = &\begin{cases} 2 & (n=0) \\ 2m-4 & (n=1) \\ \varphi(n) + (m-4)\varphi(n-1) + (-1)^m \varphi(n-m+1) + \sum_{k=2}^{m-1} (-1)^k (k+1) \varphi(n-k) & (n \geq 2). \end{cases} \end{aligned}$$

Theorem 3. *There is an augumentation map* $\epsilon : \mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R)) \to R$ *as an R-algebra homomorphism such that the Kernel* $\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R))$ *of* ϵ *satisfies*

$$\overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) \cdot \overline{\mathrm{HH}^*}(\mathrm{N}_m(R), \mathrm{N}_m(R)) = 0.$$

In particular, $\mathrm{HH}^*(\mathrm{N}_m(R), \mathrm{N}_m(R))$ *is an infinitely generated algebra over R.*

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On two-sided Harada rings

Yoshitomo Baba

In [6] M. Harada studied a left artinian ring R such that every non-small left R-module contains a non-zero injective submodule. And in [7] K. Oshiro called the ring a left Harada ring (abbreviated left H-ring). We can see many results on left Harada rings in [5] and many equivalent conditions in [1, Theorem B]. In this talk, we characterize two-sided Harada rings.

First new concepts "co-H-sequence", "H-epimorphism" and weak co-H-sequenceand are induced and by these we characterized two-sided Harada rings. Next, from a given QF ring, we construct two-sided Harada rings. And last we show that every indecomposable two-sided Harada ring which is not a Nakayama ring is constructed from a QF-ring.

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Krull–Gabriel dimension of Cohen–Macaulay modules over hypersurfaces of type (A_∞)

Naoya Hiramatsu

The notion of the Krull–Gabriel dimension has been considered under the functorial approach viewpoint of representation theory of finite dimensional algebras. It was introduced by Gabriel[3] and has been studied by many authors including Geigle[4], Krause[5] and Schröer[7].

Definition 1 (Krull Gabriel dimension)**.** Let $\mathcal{A}$ be a abelian category. Define $\mathcal{A}_{-1} = 0$. For each $n \geqq 1$, let $\mathcal{A}_n$ be the category of all objects which are finite length in $\mathcal{A}/\mathcal{A}_{n-1}$. We define KGdim $\mathcal{A} = min\{n \mid \mathcal{A} = \mathcal{A}_n\}$ if such a minimum exists, and KGdim $\mathcal{A} = \infty$ else.

The Krull–Gabriel dimension is closely related to representation types of algebras. Let A be a finite dimensional algebra and $\mathrm{mod}(A)$ a category of finitely generated A-modules. We denote by $\mathrm{mod}(\mathrm{mod}(A))$ a functor category of $\mathrm{mod}(A)$, that is, the category of finitely presented contravariant additive functors from $\mathrm{mod}(A)$ to the category of Abelian groups. It was proved by Auslander[1] that A is of finite representation type if and only if KGdim $\mathrm{mod}(\mathrm{mod}(A)) = 0$. Krause[5] shows that there are no algebras such that KGdim $\mathrm{mod}(\mathrm{mod}(A)) \neq 1$, and Geigle[4] shows that every tame hereditary algebra is of Krull–Gabriel dimension 2. Geigle[4] also shows that an algebra which is of wild representation type has Krull–Gabriel dimension ∞.

Let R be a commutative Cohen–Macaulay local ring and $\mathcal{C}(R)$ the category of maximal Cohen–Macaulay R-modules. In this paper we study the Krull–Gabriel dimension of $\underline{\mathrm{mod}}(\mathcal{C}(R))$; the full subcategory of $\mathrm{mod}(\mathcal{C}(R))$ consisting of all functors with $F(R) = 0$.

Theorem 2. *Let R be a complete Cohen–Macaulay local ring. Then R is of finite representation type if and only if* KGdim $\underline{\mathrm{mod}}(\mathcal{C}(R)) = 0$.

Let k be an algebraically closed uncountable field of characteristic not two. Next we investigate the case where R is a hypersurface of type (A_∞), that is, R is isomorphic to the ring $k[\![x_0, x_1, x_2, \ldots, x_n]\!]/(f)$, where $f = x_1^2 + x_2^2 + \cdots + x_n^2$. It is known that R is of countable CM representation type [2].

Theorem 3. *Let k be an algebraically closed uncountable field of characteristic not two. Let R be a hypersurface of type (A_∞). Then* KGdim $\underline{\mathrm{mod}}(\mathcal{C}(R)) = 2$.

The study of the Krull–Gabriel dimension of maximal Cohen–Macaulay modules over a one-dimensional hypersurface of type (A_∞) is given by Puninski[6]. His study investigates the Krull–Gabriel dimension of the definable category of maximal Cohen–Macaulay modules in $\mathrm{Mod}(R)$, so that our studies is different from his results.

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Algebraic stability theorem for derived categories of zigzag persistence modules

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This is based on the paper arXiv:2006.06924[7], which is an interaction between representation theory of algebras and topological data analysis, particularly the robustness for noises of data

Persistent homology [6] is one of the leading tools in topological data analysis. It provides a multi-scale analysis of the topological features of a given data set with the so-called persistence diagram as its output. Unlike ordinary homology, it is significant that a stability theorem holds for persistent homology [5].

The algebraic structure of persistent homology is expressed using the notion of *persistence modules*, which are representations of an equioriented A_n-type quiver [3]. This allows for an algebraic generalization of the stability theorem, which is called an algebraic stability theorem (AST; see [4], [1]). Namely, the AST guarantees that the persistence diagram is robust to changes in the given persistence module.

Moreover, a *zigzag persistence module* [3] (a representation of an A_n-type quiver with arbitrary orientation) can be applied to address several situations (e.g. time-series data) which are not covered by the theory of ordinary persistence module. Our motivation is to derive an AST for zigzag persistence modules. Botnan and Lesnick proved such a theorem by embedding the category of zigzag persistence modules into that of 2D block decomposable persistence modules [2]. Here, we adopt a different approach: We study distances on zigzag persistence modules from the viewpoint of derived categories and Auslander–Reiten quivers. For two persistence modules M, N, we can define the *interleaving distance* d_I between M and N. We denote by $\mathcal{B}(M)$ the *persistence diagram* of M, which consists of the indecomposable representations (we call them *intervals*) in the indecomposable decomposition of M. Then, the interleaving distance induces the *bottleneck distance* d_B between $\mathcal{B}(M)$ and $\mathcal{B}(N)$. Comparing these distances, the following holds.

Theorem 1 (AST [4],[1]). $\mathrm{d}_B(\mathcal{B}(M), \mathcal{B}(N)) \leq \mathrm{d}_I(M, N)$.

The distances $\mathrm{d}_I, \mathrm{d}_B$ can be extended to the derived setting $\mathrm{d}_I^D, \mathrm{d}_B^D$. Thus, we obtain the following main theorem for objects $X^\bullet, Y^\bullet$ of the derived category of persistence modules.

Theorem 2 (Derived AST [7]). $\mathrm{d}_B^D(\mathcal{B}^D(X^\bullet), \mathcal{B}^D(Y^\bullet) \leq \mathrm{d}_I^D(X^\bullet, Y^\bullet)$.

The derived category of persistence modules is triangle equivalent to that of zigzag persistence modules, depending on a classical tilting module. Through this derived equivalence, we define and compute distances on the derived category of zigzag persistence modules and proved an algebraic stability theorem.

Corollary 3 ([7]). *For the derived category of zigzag persistence modules, an AST holds.*

As a consequence, an AST holds for zigzag persistence modules.

Finally, we also compare our distance with the distance for purely zigzag persistence modules introduced by Botnan–Lesnick and the sheaf-theoretic convolution distance due to Kashiwara–Schapira.

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