Faces of certain neighborhoods of presilting cones

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Background

Let A be a fin. dim. algebra over a field K.

- 2-psilt $A := \{$basic 2-term presilting cpx in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)\}/\cong$.
- In the study of 2-psilt A, the Grothendieck group $K_0(\operatorname{proj} A)$ naturally appears.
- For each $U = \bigoplus_{i=1}^m U_i \in$ 2-psilt A with U_i indec., we define silting cone in $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by $C^\circ(U) := \sum_{i=1}^m \mathbb{R}_{>0}[U_i]$.
- Silting cones are TF equiv. classes defined by semistable torsion pairs of [Baumann-Kamnitzer-Tingley].
- It is sometimes useful to consider 2-psilt$_U A := \{V \in$ 2-psilt $A \mid U \in \operatorname{add} V\}$.
- To relate 2-psilt$_U A$ and TF equivalence, I (Asai) introduced the interval neighborhood $N_U \supset C^\circ(U)$.
- Its closure $\overline{N_U}$ is a rational polyhedral cone in $K_0(\operatorname{proj} A)_{\mathbb{R}}$.
- Today, we talk on the faces of $\overline{N_U}$.

Setting

Let A be a fin. dim. algebra over a field K.

- proj A: the category of fin. gen. projective A-modules.
- mod A: the category of fin. dim. A-modules.
- $K_0(C)_{\mathbb{R}} := K_0(C) \otimes_{\mathbb{Z}} \mathbb{R}$: the real Grothendieck group.
- $K_0(\operatorname{proj} A)_{\mathbb{R}} = \bigoplus_{i=1}^{m} \mathbb{R}[P_i] \cong \mathbb{R}^n$.
  - $P_1, \dots, P_n$: the non-iso. indec. proj. A-modules.
- $K_0(\operatorname{mod} A)_{\mathbb{R}} = \bigoplus_{i=1}^{m} \mathbb{R}[L_i] \cong \mathbb{R}^n$.
  - $L_1, \dots, L_n$: the non-iso. simple A-modules.
  - L_i is the simple top of P_i.
- Each $\theta = \sum_{i=1}^{n} a_i[P_i] \in K_0(\operatorname{proj} A)_{\mathbb{R}}$ gives an $\mathbb{R}$-linear map

$$\theta\colon K_0(\operatorname{mod} A)_{\mathbb{R}} \to \mathbb{R};$$
$$\sum_{i=1}^{n} b_i[L_i] \mapsto \sum_{i=1}^{n} a_i b_i \dim_K \operatorname{End}_A(L_i).$$

Presilting complexes

Definition [Keller-Vossieck, Adachi-Iyama-Reiten]

Let $U = (U^{-1} \to U^0) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ be a 2-term complex.

(1) U: 2-term presilting $:\Longleftrightarrow$ $\operatorname{Hom}_{\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)}(U, U[>0]) = 0$.

(2) U: 2-term silting $:\Longleftrightarrow$ U: 2-term presilting, $|U| = n$.

2-psilt $A := \{$basic 2-term presilting complexes$\}/\cong$.
2-silt $A := \{$basic 2-term silting complexes$\}/\cong$.

Definition

Let $U = \bigoplus_{i=1}^m U_i \in$ 2-psilt A with U_i: indec.
We define the silting cones $C^\circ(U), C(U)$ in $K_0(\operatorname{proj} A)_{\mathbb{R}}$ by

$$C^\circ(U) := \sum_{i=1}^m \mathbb{R}_{>0}[U_i], \quad C(U) := \sum_{i=1}^m \mathbb{R}_{\geq 0}[U_i].$$

TF equivalence

Definition [(1) Baumann-Kamnitzer-Tingley, (2) King]

(1) We define the semistable torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta)$ and $(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ by

$$\overline{\mathcal{T}}_\theta := \{M \in \operatorname{mod} A \mid \theta(N) \geq 0 \text{ for any quotient } N \text{ of } M\},$$
$$\mathcal{F}_\theta := \{M \in \operatorname{mod} A \mid \theta(L) < 0 \text{ for any submodule } L \neq 0 \text{ of } M\},$$
$$\mathcal{T}_\theta := \{M \in \operatorname{mod} A \mid \theta(N) > 0 \text{ for any quotient } N \neq 0 \text{ of } M\},$$
$$\overline{\mathcal{F}}_\theta := \{M \in \operatorname{mod} A \mid \theta(L) \leq 0 \text{ for any submodule } L \text{ of } M\}.$$

(2) $\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$: the θ-semistable subcategory.
$\mathcal{W}_\theta$ is a wide subcat., so $[\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta]$ is a wide interval.

Definition

$\theta, \theta' \in K_0(\operatorname{proj} A)_\mathbb{R}$ are TF equivalent $:\Longleftrightarrow$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = (\overline{\mathcal{T}}_{\theta'}, \mathcal{F}_{\theta'}), \quad (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\mathcal{T}_{\theta'}, \overline{\mathcal{F}}_{\theta'}).$$

Presilting cones are TF equiv. classes

Proposition [$\Rightarrow$: Yurikusa, Brüstle-Smith-Treffinger, $\Leftarrow$: A]

Let $U \in$ 2-psilt A.
Then, $C^\circ(U)$ is a TF equiv. class such that $\theta \in C^\circ(U) \iff$

$$(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta) = ({}^\perp H^{-1}(\nu U), \operatorname{Sub} H^{-1}(\nu U)),$$
$$(\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta) = (\operatorname{Fac} H^0(U), H^0(U)^\perp).$$

$H^0(U)$ is a τ-rigid module, $H^{-1}(\nu U)$ is a τ^{-1}-rigid module.

Definition

For any $U \in$ 2-psilt A, we set

$$(\overline{\mathcal{T}}_U, \mathcal{F}_U) := ({}^\perp H^{-1}(\nu U), \operatorname{Sub} H^{-1}(\nu U)),$$
$$(\mathcal{T}_U, \overline{\mathcal{F}}_U) := (\operatorname{Fac} H^0(U), H^0(U)^\perp).$$

$[\mathcal{T}_U, \overline{\mathcal{T}}_U]$ is a wide interval.

Interval neighborhoods of silting cones

We want to study $\text{2-psilt}_U\, A := \{V \in \text{2-psilt}\, A \mid U \in \operatorname{add} V\}$.

Definition

For any $U \in \text{2-psilt}\, A$, set the interval neighborhood $N_U \supset C^\circ(U)$ by

$$N_U := \{\theta \in K_0(\operatorname{proj} A)_\mathbb{R} \mid H^0(U) \in \mathcal{T}_\theta,\ H^{-1}(\nu U) \in \mathcal{F}_\theta\}$$
$$= \{\theta \in K_0(\operatorname{proj} A)_\mathbb{R} \mid [\mathcal{T}_\theta, \overline{\mathcal{T}}_\theta] \subset [\mathcal{T}_U, \overline{\mathcal{T}}_U]\}.$$

Lemma

Let $U, V \in \text{2-psilt}\, A$.

(1) N_U is an open neighborhood of $C^\circ(U)$.

(2) N_U is given by fin. many linear strict inequalties.

(3) $U \in \operatorname{add} V \iff [\mathcal{T}_V, \overline{\mathcal{T}}_V] \subset [\mathcal{T}_U, \overline{\mathcal{T}}_U]$
$\iff C^\circ(V) \subset N_U \iff N_V \subset N_U$.

The minimality of interval neighborhoods

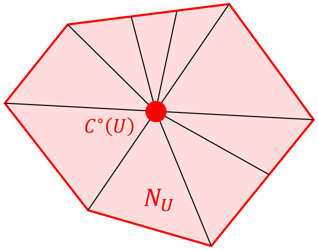
Lemma

Let $U \in$ 2-psilt A.

Then, N_U is the smallest set satisfying

(a) N_U is a neighborhood of $C^\circ(U)$;

(b) N_U is a union of TF equiv. classes.



Each black line & light red triangle is a TF eq. class.
The red lines are the boundary of N_U,
so they don't belong to N_U, since N_U is open.

The closure of interval neighborhoods

We also focus on the closure $\overline{N_U} \subset K_0(\text{proj}\, A)_{\mathbb{R}}$.

Lemma

Let $U, V \in$ 2-psilt A.

(1) $\overline{N_U} = \{\theta \in K_0(\text{proj}\, A)_{\mathbb{R}} \mid H^0(U) \in \overline{\mathcal{T}}_\theta,\ H^{-1}(\nu U) \in \overline{\mathcal{F}}_\theta\}$.
In particular, $\overline{N_U}$ is a union of TF equiv. classes.

(2) $\overline{N_U} \supset C(U)$.

(3) $\overline{N_U}$ is a rational polyhedral cone in $K_0(\text{proj}\, A)_{\mathbb{R}}$.

(4) $U \oplus V$: presilting $\iff N_U \cap N_V \neq \emptyset \iff C(V) \subset \overline{N_U}$.
In this case, $N_{U \oplus V} = N_U \cap N_V$.

By (3), we can consider the faces of $\overline{N_U}$.

Faces of $\overline{N_U}$

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i indec.
For any $I \subset \{1, 2, \ldots, m\}$, we set $U_I := \bigoplus_{i \in I} U_i$.

Definition-Proposition [A-Iyama]

Let $F \in \mathsf{Face}\, \overline{N_U}$.

(1) $F \cap C(U) = C(U/U_{I_F})$, where $I_F := \{i \in \{1, \ldots, m\} \mid [U_i] \notin F\}$.

(2) If $\dim_{\mathbb{R}} F = n - 1$, then $\# I_F = 1$.

(3) For any $I \subset \{1, 2, \ldots, m\}$, we define

$$\mathsf{Face}_I^{\times} \overline{N_U} := \{F \in \mathsf{Face}\, \overline{N_U} \mid I_F = I\}.$$

Then, we have a (not necessarily convex) subset

$$\partial_I := \bigcup_{F \in \mathsf{Face}_I^{\times} \overline{N_U}} F = \overline{N_U} \setminus \bigcup_{i \in I} N_{U_i} \subset \overline{N_U}.$$

N_U and τ-tilting reduction

Fix $U \in$ 2-psilt A, and consider 2-psilt$_U$ A.
Take $S \in$ 2-silt A such that $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U$, and set $B := \mathrm{End}_A(H^0(S))/\langle e \rangle$, where e is the idempotent $H^0(S) \to H^0(U) \to H^0(S)$.

Proposition [Jasso]

(1) There exists a cat. eq. $\Phi \colon \mathcal{W}_U := \overline{\mathcal{T}}_U \cap \overline{\mathcal{F}}_U \to \mathrm{mod}\, B$.

(2) There exists a bijection red: 2-psilt$_U$ $A \to$ 2-psilt B.

Proposition [A]

There exists a linear map $\pi \colon K_0(\mathrm{proj}\, A)_{\mathbb{R}} \to K_0(\mathrm{proj}\, B)_{\mathbb{R}}$ such that

(a) The restriction $\pi|_{N_U}$ is surjective, and $\mathrm{Ker}\, \pi = \mathbb{R}C(U)$.

- If $S = U \oplus \bigoplus_{j=m+1}^{n} S_j$, then $\pi([S_j]) = [P^B_{j-m}] \in K_0(\mathrm{proj}\, B)_{\mathbb{R}}$.

(b) $\forall \theta \in N_U$, $\Phi(\mathcal{T}_\theta \cap \mathcal{W}_U) = \mathcal{T}_{\pi(\theta)}$, $\Phi(\mathcal{F}_\theta \cap \mathcal{W}_U) = \mathcal{F}_{\pi(\theta)}$.

- {TF eq. classes in N_U} $\cong$ {TF eq. classes in $K_0(\mathrm{proj}\, B)_{\mathbb{R}}$}.

(c) $\forall V \in$ 2-psilt$_U$ A, $\pi(C^\circ(V)) = C^\circ(\mathrm{red}(V))$.

Main result 1

Theorem 1 [A-Iyama]

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i indec., and $I \subset \{1, 2, \ldots, m\}$.
Set $\Sigma_I := \{\pi(F) \mid F \in \mathrm{Face}_I^{\times} \overline{N_U}\}$.

(1) π induces a bijection

$$\begin{aligned} \mathrm{Face}_I^{\times} \overline{N_U} &\to \Sigma_I \\ F &\mapsto \pi(F) \\ \pi^{-1}(\sigma) \cap \partial_I &\leftarrow\!\shortmid \sigma. \end{aligned}$$

(2) $\forall F \in \mathrm{Face}_I^{\times} \overline{N_U}$, $\dim_{\mathbb{R}} \pi(F) = \dim_{\mathbb{R}} F - \#I$.

(3) Σ_I is a finite rational polyhedral fan covering $K_0(\mathrm{proj}\, B)_{\mathbb{R}}$;

- **(a)** $\forall \sigma \in \Sigma_I$, $\mathrm{Face}\, \sigma \subset \Sigma_I$.
- **(b)** $\forall \sigma_1, \forall \sigma_2 \in \Sigma_I$, $\sigma_1 \cap \sigma_2 \in (\mathrm{Face}\, \sigma_1) \cap (\mathrm{Face}\, \sigma_2)$.

M-TF equivalnce

For any $M \in \operatorname{mod} A$ and $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$,
we have unique short exact sequences

$$0 \to \bar{\mathrm{t}}_\theta M \to M \to \mathrm{f}_\theta M \to 0 \quad (\bar{\mathrm{t}}_\theta M \in \overline{\mathcal{T}}_\theta,\ \mathrm{f}_\theta M \in \mathcal{F}_\theta),$$
$$0 \to \mathrm{t}_\theta M \to M \to \bar{\mathrm{f}}_\theta M \to 0 \quad (\mathrm{t}_\theta M \in \mathcal{T}_\theta,\ \bar{\mathrm{f}}_\theta M \in \overline{\mathcal{F}}_\theta).$$

Moreover, we set $\mathrm{w}_\theta M := \bar{\mathrm{t}}_\theta M / \mathrm{t}_\theta M \in \mathcal{W}_\theta = \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta$.

Definition (cf. [Aoki-Higashitani-Iyama-Kase-Mizuno])

Let $M \in \operatorname{mod} A$, and $\theta, \theta' \in K_0(\operatorname{proj} A)_{\mathbb{R}}$.
Then, we say θ, θ' are M-TF equivalent if

- **(a)** $\mathrm{t}_\theta M = \mathrm{t}_{\theta'} M$ and $\mathrm{w}_\theta M = \mathrm{w}_{\theta'} M$ and $\mathrm{f}_\theta M = \mathrm{f}_{\theta'} M$; and
- **(b)** The compos. factors of $\mathrm{w}_\theta M = \mathrm{w}_{\theta'} M$ in $\mathcal{W}_\theta$ and $\mathcal{W}_{\theta'}$ coincide.

We set $\Sigma(M) :=$ {the closures of all M-TF equiv. classes}.
$\Sigma(M)$ is a finite rational polyhedral fan covering $K_0(\operatorname{proj} A)_{\mathbb{R}}$.

Main result 2

Theorem 2 [A-Iyama]

Let $U = \bigoplus_{i=1}^{m} U_i \in$ 2-psilt A with U_i indec.
Then, there exist $M_1, M_2, \ldots, M_m \in$ mod B such that

$$\forall I \subset \{1, 2, \ldots, m\}, \quad \Sigma\left(\bigoplus_{i \in I} M_i\right) = \Sigma_I.$$

- Take the max. and min. completions $S, T \in$ 2-silt A of U.
  - $\overline{\mathcal{T}}_S = \overline{\mathcal{T}}_U, \overline{\mathcal{F}}_T = \overline{\mathcal{F}}_U$.
  - $S = \bigoplus_{i=1}^{n} S_i, T = \bigoplus_{i=1}^{n} T_i$ with $S_i = T_i = U_i$ for $i \in \{1, \ldots, m\}$.
- Take the 2-term simple-minded collections $\mathcal{X} = (X_i)_{i=1}^{n}$ and $\mathcal{Y} = (Y_i)_{i=1}^{n}$ corresponding to S and T.
- We have the triangle $W_i \to Y_i \to X_i \to W_i[1]$ with $W_i \in \mathcal{W}_U$.
- Set $M_i := \Phi(W_i) \in$ mod B.

Thank you for your attention.

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