The correspondence between silting objects and t-structures for non-positive dg algebras

via Koszul duality

Riku Fushimi

Nagoya University
fushimi.riku.h9@s.mail.nagoya-u.ac.jp

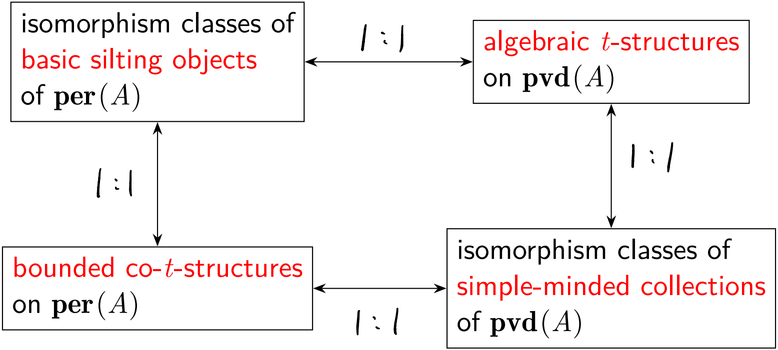
17 September, 2024

Main theorem

Let A be a dg $\Bbbk$-algebra such that

- (locally finite) $H^i A$ is finite dimensional for every $i \in \mathbb{Z}$,
- (non-positive) $H^i A = 0$ for every $i \in \mathbb{Z}_{>0}$.

Then



Conventions

- $\Bbbk$ is a field.
- All (dg) algebras and categories are defined over $\Bbbk$.
- All subcategoires are full, additive, and closed under taking direct summunds.
- $\mathcal{T}$ is a triangualted category.
- $\mathbf{add}\, X$ is the smallest subcategory containing X.
- $\mathbf{Filt}\, X$ is the smallest subcategory containing X and closed under taking extensions.
- $\mathbf{thick}\, X$ is the smallest thick subcategory containing X.

R. Fushimi, *The correspondence between silting objects and t-structures for non-positive dg algebras*, arXiv:2312.17597.
R. Fushimi, *Contravariant Koszul duality between non-positive and positive dg algebras*, arXiv:2409.08842.

Table of Contents

Table of Contents

Differential graded algebras

Definition

A pair (A, d) of a $\mathbb{Z}$-graded algebra A and a linear map $d\colon A \to A$ is differential graded (dg) algebra if

- $d(A^i) \subseteq A^{i+1}$,
- $d^2 = 0$,
- $d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b)$.

Example

Any algebra can be regarded as a dg algebra concentrated in degree 0.

Example

Let Λ be an algebra and $M \in \mathbf{D}(\Lambda)$. Then $\mathbf{R}\mathrm{End}_\Lambda(M) := \mathbf{R}\mathrm{Hom}_\Lambda(M, M)$ has a natural dg algebra structure.

(Non-)positive dg algebras

Definition

Let A be a dg algebra.

- A is called locally finite (l.f.) if $H^i A$ is finite-dimensional for every $i \in \mathbb{Z}$.
- A is called non-positive if $H^{>0} A = 0$.
- A is called positive if $H^{<0} A = 0$ and $H^0 A$ is semisimple.

Example

- Let Λ be a finite-dimensional algebra. Then
  - (i) Λ is a l.f. non-positive dg algebra,
  - (ii) $\mathbf{R}\mathrm{End}_\Lambda(\mathrm{top}\,\Lambda)$ is a l.f. positive dg algebra.
- Let (Q, W) be a Jacobi-finite quiver with potential. Then the complete Ginzburg dg algebra $\widehat{\Gamma}(Q, W)$ is l.f. non-positive dg algebra.

For a dg algebra A, we can define the derived category $\mathbf{D}(A)$ of A as in the case of ordinary algebras.

Definition

Let A be a dg algebra.

- $\mathbf{per}(A) := \mathbf{thick}_{\mathbf{D}(A)} A$ is called the perfect derived category of A.
- $\mathbf{pvd}(A) := \{M \in \mathbf{D}(A) \mid H^*M \text{ is finite-dimensional}\}$ is called the perfectly valued derived category of A.

Example

If $A = \Lambda$ be a finite-dimensional algebra, then we have $\mathbf{per}(\Lambda) = \mathbf{K}^b(\mathbf{proj}\Lambda)$ and $\mathbf{pvd}(\Lambda) = \mathbf{D}^b(\mathbf{mod}\Lambda)$.

Table of Contents

Silting objects

Definition

A object $M \in \mathcal{T}$ is called a silting object if

- $\mathcal{T}(M, M[>0]) = 0$,
- $\mathbf{thick}\, M = \mathcal{T}$.

A silting object $M \in \mathcal{T}$ is called a tilting object if

- $\mathcal{T}(M, M[<0]) = 0$.

Remark (Aihara-Iyama)

Tilting objects cannot always be mutated , but silting objects can always be mutated.

Example

A: non-positive dg algebra $\Rightarrow$ A is a silting object of $\mathbf{per}(A)$.

Bounded co-t-structures

Definition

A pair of subcategories $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$ is a (bounded) co-t-structure on $\mathcal{T}$ if

- $\mathcal{T}_{\leq 0}[1] \subseteq \mathcal{T}_{\leq 0}$,
- $\mathcal{T}(X, Y[1]) = 0$ for every $X \in \mathcal{T}_{\geq 0}$ and $Y \in \mathcal{T}_{\leq 0}$,
- $\mathcal{T}_{\geq 0} * (\mathcal{T}_{\leq 0}[1]) = \mathcal{T}$,
- (bounded) $\mathbf{thick}\,\mathcal{T}_0 = \mathcal{T}$.

$\mathcal{T}_0 := \mathcal{T}_{\geq 0} \cap \mathcal{T}_{\leq 0}$ is called the co-heart of $(\mathcal{T}_{\geq 0}, \mathcal{T}_{\leq 0})$.

Fact (Mendoza-Sáenz-Santiago-Souto Salonia.)

Let A be a l.f. non-positive dg algebra. Then

isomorphism classes of basic silting objects of $\mathbf{per}(A)$	$\overset{1:1}{\longleftrightarrow}$	bounded co-t-structures on $\mathbf{per}(A)$

s.t. the coheart of the co-t-structure corresponding to M is $\mathbf{add}\,M$.

Simple-minded collections

Definition

A set of objects $\mathcal{L} \subseteq \mathcal{T}$ is called a simple-minded collection (SMC) if

- $\mathcal{T}(L_1, L_2) = \begin{cases} \text{division ring} & \text{if } L_1 = L_2 \\ 0 & \text{otherwise,} \end{cases} \qquad (L_i \in \mathcal{L})$
- $\mathcal{T}(\mathcal{L}, \mathcal{L}[< 0]) = 0,$
- $\mathbf{thick}\mathcal{L} = \mathcal{T}.$

Remark

SMCs were first studied by Rickard in the context of derived equivalences of symmetric algebras.

Example

Let A be a l.f. non-positive dg algebra. Then the set of simple H^0A-modules is a SMC of $\mathbf{pvd}(A)$.

Algebraic t-structures

Definition

A pair of subcategories $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called an (algebraic) t-structure if

- $\mathcal{T}^{\leq 0}[1] \subseteq \mathcal{T}^{\leq 0}$,
- $\mathcal{T}(X[1], Y) = 0$ for every $X \in \mathcal{T}^{\leq 0}$ and $Y \in \mathcal{T}^{\geq 0}$,
- $(\mathcal{T}^{\leq 0}[1]) * \mathcal{T}^{\geq 0} = \mathcal{T}$,
- (algebraic) $\mathcal{T}^0$ is length and $\mathbf{thick}\mathcal{T}^0 = \mathcal{T}$.

$\mathcal{T}^0 := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is called the heart of $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$.

Fact (Deligne-Beilinson-Bernstein)

Let A be a l.f. non-positive dg algebra. Then

isomorphism classes of SMC of $\mathbf{pvd}(A)$	$\overset{1:1}{\longleftrightarrow}$	algebraic t-structures on $\mathbf{pvd}(A)$

s.t. the heart of the t-structure corresponding to $\mathcal{L}$ is $\mathbf{Filt}\mathcal{L}$.

Table of Contents

ST-correspondence

Theorem (Keller-Vossieck, Koenig-Yang, Keller-Nicolas, Su-Yang, F) (ST-correspondence)

Let A be a l.f. non-positive dg algebra. Then the map

$$\left\{\begin{array}{l}\text{isomorphism classes of}\\ \text{basic silting objects of } \mathbf{per}(A)\end{array}\right\} \xrightarrow{1:1} \left\{\begin{array}{l}\text{algebraic } t\text{-structures}\\ \text{on } \mathbf{pvd}(A)\end{array}\right\}$$

that maps M to $(\mathbf{pvd}(A)^{\leq 0}_{M}, \mathbf{pvd}(A)^{\geq 0}_{M})$ is bijective, where

$$\mathbf{pvd}(A)^{\leq 0}_{M} := \{X \in \mathbf{pvd}(A) \mid \mathbf{D}(A)(M, X[>0]) = 0\},$$
$$\mathbf{pvd}(A)^{\geq 0}_{M} := \{X \in \mathbf{pvd}(A) \mid \mathbf{D}(A)(M, X[<0]) = 0\}.$$

Example

If $A = \Lambda$ is a finite-dimensional algebra, then $(\mathbf{pvd}(\Lambda)^{\leq 0}_{\Lambda}, \mathbf{pvd}(\Lambda)^{\geq 0}_{\Lambda})$ is the standard t-structure on $\mathbf{pvd}(\Lambda) = \mathbf{D}^b(\mathbf{mod}\Lambda)$.

ST-correspondence

Corollary

Let A be a l.f. non-positive dg algebra. The following sets correspond bijectively:

- $\{$iso classes of basic silting objects of $\mathbf{per}(A)\}$,
- $\{$bounded co-t-structures on $\mathbf{per}(A)\}$,
- $\{$iso classes of SMCs of $\mathbf{pvd}(A)\}$,
- $\{$algebraic t-structures on $\mathbf{pvd}(A)\}$.

Table of Contents

Kosul duals

The main ingredient in proving ST-correspondence is the Koszul duality for dg algebras introduced by Keller.

Definition

- Let A be a l.f. non-positive dg algebra. Then we put $A^! := \mathbf{R}\mathrm{End}_A(\operatorname{top} H^0 A)$.
- Let A be a l.f. positive dg algebra. Then we put $A^! := \mathbf{R}\mathrm{End}_A(H^0 A)$.

In the following, we say a l.f. positive dg algebra A is nice if $\mathbf{Filt}\, A$ has a projective generator.

Theorem (Lu-Palmieri-Wu-Zhang, F) (Koszul duality)

- Let A be a l.f. non-positive dg algebra. Then $A^!$ is a l.f. nice positive dg algebra, and $\mathbf{R}\mathrm{Hom}_A(-, \mathrm{top}\, H^0 A)$ induces

$$\begin{array}{ccc} \mathbf{per}(A) & \xrightarrow{\simeq} & \mathbf{pvd}((A^!)^{\mathrm{op}})^{\mathrm{op}} \\ \cap & & \cap \\ \mathbf{D}^-_{\mathrm{fd}}(A) & \xrightarrow{\simeq} & \mathbf{D}^+_{\mathrm{fd}}((A^!)^{\mathrm{op}})^{\mathrm{op}} \\ \cup & & \cup \\ \mathbf{pvd}(A) & \xrightarrow{\simeq} & \mathbf{per}((A^!)^{\mathrm{op}})^{\mathrm{op}}. \end{array}$$

- Let A be a l.f. nice positive dg algebra. Then $A^!$ is a l.f. non-positive dg algebra, and $\mathbf{R}\mathrm{Hom}_A(-, H^0 A)$ induces

$$\begin{array}{ccc} \mathbf{per}(A) & \xrightarrow{\simeq} & \mathbf{pvd}((A^!)^{\mathrm{op}})^{\mathrm{op}} \\ \cap & & \cap \\ \mathbf{D}^+_{\mathrm{fd}}(A) & \xrightarrow{\simeq} & \mathbf{D}^-_{\mathrm{fd}}((A^!)^{\mathrm{op}})^{\mathrm{op}} \\ \cup & & \cup \\ \mathbf{pvd}(A) & \xrightarrow{\simeq} & \mathbf{per}((A^!)^{\mathrm{op}})^{\mathrm{op}}. \end{array}$$

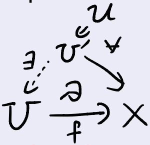
Table of Contents

Functorially finite subcategories

Definition

Let $\mathcal{U}$ be a subcategory of $\mathcal{T}$. Let $U \in \mathcal{U}$ and $X \in \mathcal{T}$. A morphism $f\colon U \to X$ is called a right $\mathcal{U}$-approximation of X if

$$f \circ - \colon \mathcal{T}(U', U) \to \mathcal{T}(U', X)$$



is surjective for every $U' \in \mathcal{U}$. Dually, we define left $\mathcal{U}$-approximations.

Definition

- A subcategory $\mathcal{U} \subseteq \mathcal{T}$ is called contravariantly finite if every $X \in \mathcal{T}$ has a right $\mathcal{U}$-approximation. Dually, we define covariantly finite subcategories.
- A subcategory $\mathcal{U} \subseteq \mathcal{T}$ is called functorially finite if $\mathcal{U}$ is contravariantly finite and covariantly finite.

Functorially finite hearts

A t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is called bounded if $\mathbf{thick}\,\mathcal{T}^0 = \mathcal{T}$.

Theorem (Coelho Simões-Pauksztello-Ploog, F)

$\mathcal{T}$: Hom-finite algebraic triangulated category.
$\mathcal{L}$: SMC of $\mathcal{T}$ s.t. $\mathbf{Filt}\,\mathcal{L}$ has a projective generator.

(1) $\mathcal{T} \simeq \mathbf{pvd}(A)$ for some l.f. non-positive dg algebra A.

(2) For a heart $\mathcal{H}$ of a bounded t-structure on $\mathcal{T}$, the following conditions are equivalent:

- (i) $\mathcal{H}$ is contravariantly finite in $\mathcal{T}$,
- (i)' $\mathcal{H}$ is covariantly finite in $\mathcal{T}$,
- (ii) $\mathcal{H}$ is functorially finite in $\mathcal{T}$,
- (iii) $\mathcal{H}$ is a length category,
- (iv) $\mathcal{H}$ has a projective generator,
- (v) $\mathcal{H} \simeq \mathbf{mod}\,\Lambda$ for some finite-dimensional algebra Λ.

References I

[AI] T. Aihara, O. Iyama, *Silting mutation in triangulated categories*, J. London. Math. Soc. (2) **85** (2012), no. 3, 633–668.

[BBD] P. Deligne, A. Beilinson, J. Bernstein, *Faisceaux pervers*, Astérisque, Vol.100, (1983), 1-172.

[F1] R. Fushimi, *The correspondence between silting objects and t-structures for non-positive dg algebras*, arXiv:2312.17597.

[F2] R. Fushimi, *Contravariant Koszul duality between non-positive and positive dg algebras*, arXiv:2409.08842.

[Ke] B. Keller, *Deriving DG categories*, Ann. Sci. Ecole Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[KeN1] B. Keller, P. Nicolás, *Weight structures and simple dg modules for positive dg algebras*, Int. Math. Res. Not. IMRN 2013, no. 5, 1028–1078.

[KeN2] B. Keller, P. Nicolás, *Cluster hearts and cluster tilting objects*, work in preparation.

[KeV] B. Keller, D. Vossieck, *Aisles in derived categories*, Bull. Soc. Math. Belg. Sér. A **40** (1988), no. 2, 239–253.

[KoY] S. Koenig, D. Yang, *Silting objects, simple-minded collections, t-structures and co-t-structures for finite-length algebras*, Doc. Math. **19** (2014), 403–438.

References II

[LPWZ] D. M. Lu, J. H. Palmieri, Q. S. Wu, and J. J. Zhang.*Koszul equivalences in A_∞-algebras*, New York J. Math, 14:325–378, 2008.

[MSSS] O. Mendoza, E. C. S´aenz, V. Santiago, M. J. Souto Salonia. *Auslander–Buchweitz Context and Co-tstructures*, Applied Categorical Structures, vol.21, 417–440 (2013).

[R] J. Rickard, *Equivalences of derived categories for symmetric algebras*, Journal of pure and applied Algebra 61.3 (1989): 303-317.

[SPP] R. Coelho Simões, D. Pauksztello, and D. Ploog, *Functorially finite hearts, simple-minded systems in negative cluster categories, and noncrossing partitions*, Compositio Mathematica 158.1 (2022): 211-243.

[SY] H. Su, D. Yang, *From simple-minded collections to silting objects via Koszul duality*. Algebras and Representation Theory, vol.22, 219–238 (2019).

Thank you for your attention!