Semibricks and Spherical objects

Wahei Hara (Kavli IPMU)
joint work with M. Wemyss

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Statement of Main Theorem

Today we will discuss

semibrick complexes over silting discrete algebra

Let A be a finite dimensional $\mathbb{C}$-algebra.

Definition

$S \in \mathrm{D}^{\mathrm{b}}(A) := \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A)$.

- S is brick if $\mathrm{Ext}^i(S, S) = 0$ for $i < 0$ and $\mathrm{End}(S) = \mathbb{C}$.
- S is semibrick (complex) if
  - **(a)** $\mathrm{Ext}^i(S, S) = 0$ for $i < 0$ and
  - **(b)** $S \simeq \bigoplus_i S_i$, where $\{S_i\}$ is a collection bricks with $\mathrm{Hom}(S_i, S_j) = \mathbb{C}\,\delta_{ij}$.
- S is a simple-minded collection (smc) if
  - **(a)** S is a semibrick (complex), and
  - **(b)** S generates $\mathrm{D}^{\mathrm{b}}(A)$.

(Semi)bricks are generalisation of (sum of) simple modules.

semibricks (simples) $\leftrightarrow$ presiltings (projectives)

Definition

$P \in \mathbf{Perf}(A)$.

- P is presilting if $\mathbf{Ext}^i(P, P) = 0$ for $i > 0$.
- P is silting if
  - (a) P is presilting, and
  - (b) P generates $\mathbf{Perf}(A)$.

Example

$\{S_1, \cdots, S_n\}$ is the full collection of simple modules over A.

- $S = \bigoplus_{i=1}^n S_i$ is an SMC and any nonzero summands are semibricks.
- A is silting, and all projective A-modules are presilting.

Theorem (Koenig-Yang)

There is a bijection among

- $\mathbf{silt}\, A := \{\text{siltings in } \mathbf{Perf}\, A\}/\simeq\ \subset \{\text{presiltings}\}$
- $\mathbf{smc}\, A := \{\text{smcs in } \mathbf{D}^{\mathrm{b}}(A)\}/\simeq\ \subset \{\text{semibricks}\}$
- $\{\text{bdd } t\text{-strs. on } \mathbf{D}^{\mathrm{b}}(A) \text{ with length } \heartsuit\} \subset \{\text{bdd } t\text{-strs.}\}$

KY-bij is (partial-)order-preserving.

- $P, P' \in \mathbf{silt}\, A$

$$P \geq P' :\Leftrightarrow \mathrm{Ext}^i(P, P') = 0 \text{ for all } i > 0$$

- $S, S' \in \mathbf{smc}\, A$

$$S \geq S' :\Leftrightarrow \mathrm{Ext}^i(S', S) = 0 \text{ for all } i < 0$$

E.g. $P \geq P[1]$ and $S \geq S[1]$.

Question

What kind of control exits for

$$\{\text{presiltings}\},\ \{\text{semibricks}\},\ \text{and}\ \{\text{bdd } t\text{-strs.}\}?$$

A. They behaves very well when A is silting discrete.

Definition (Aihara-Mizuno)

A is silting discrete if for any $P \in \mathbf{silt}\, A$,

$$\sharp\{Q \in \mathbf{silt}\, A \mid P \geq Q \geq P[1]\} < \infty.$$

$P \geq Q \geq P[1] \Leftrightarrow Q$ is $\mathbf{2}$-term w.r.t. ($\heartsuit$ corresponding to) P.

silting discrete "$=$" everywhere τ-tilting finite.

Let A be a silting discrete algebra.

Main Theorem (H-Wemyss)

For $x \in \mathrm{D}^{\mathrm{b}}(A)$, the following hold.

(1) $\mathbf{Ext}^{<0}(x,x) = 0$ iff $\exists\mathcal{H}$ length ♡ on $\mathrm{D}^{\mathrm{b}}(A)$ s.t. $x \in \mathcal{H}$.

(2) x is semibrick cpx. iff $\exists x'$ s.t. $x \oplus x' \in \mathbf{smc}\, A$.
(any semibrick can be completed to an smc)

Other Known Results

- (Aihara-Mizuno) P is presilting iff $\exists Q$ s.t. $P \oplus Q \in \mathbf{silt}\, A$.
- (Adachi-Mizuno-Yang, Pauksztello-Saorin-Zvonareva)
  (1) All bdd t-str have length hearts.
  (2) The space of Bridgeland stability conditions $\mathbf{Stab}\, \mathrm{D}^{\mathrm{b}}(A)$ is connected.

These results classify (semi)bricks, presiltings, and t-structures.

Examples and Contraction Algebras

Let us see examples of silting discrete algebras.

(1) Preprojective algebras of Dynkin type $A_1, A_2, D_{2n}, E_7, E_8$ are silting discrete.

- A_1 and A_2 have the small rank (easy to study).
- D_{2n}, E_7, E_8 have trivial Nakayama involution.

Question

Are all type A_n silting discrete? (especially A_3 !!)

2-term semibricks and presiltings over type A preprojective algebras are studied by

Barnard-Hanson, Mizuno, Iyama-Williams, etc.

(2) Brauer graph algebra whose Brauer graph has at most one odd length cycle and no even length cycle.

(3) (3-fold) contraction algebras (Main Example today).

Let $f\colon X \to \operatorname{Spec} R$ be a 3-fold flopping contraction s.t.

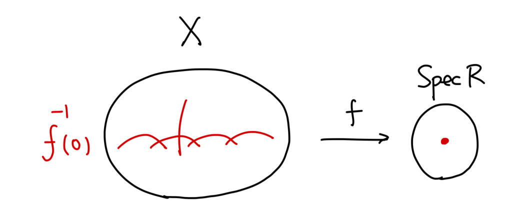
- R is complete local ($o \in \operatorname{Spec} R$ the max ideal) and
- X has at worst Gorenstein terminal singularities.

Then it is known that

(a) $f^{-1}(o)_{\mathrm{red}} = \bigcup C_i$ is a tree of projective lines $C_i \simeq \mathbb{P}^1$.

(b) $R \simeq \mathbb{C}[\![x,y,z,w]\!]/(f + wg)$ (isolated cDV singularity),

where $\begin{cases} f \in \mathbb{C}[\![x,y,z]\!] \text{ is simple ADE and} \\ g \in \mathbb{C}[\![x,y,z,w]\!] \text{ is arbitrary.} \end{cases}$



Michel Van den Bergh tells us:

- $\exists$ (canonical) NC R-algebra A such that

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\, X) \simeq \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\, A).$$

- $A \simeq \mathrm{End}_R(R \oplus M)$ for $M \in \mathrm{CM}\, R$.

Definition (Donovan-Wemyss)

The contraction algebra $A_{\mathrm{con}} = A_{\mathrm{con},f}$ associated to the contraction $f\colon X \to \mathrm{Spec}\, R$ is

$$\begin{aligned} A_{\mathrm{con},f} &:= \underline{\mathrm{End}}_R(M) \\ &= \mathrm{Hom}_R(M, M)/\{M \to P \to M \mid P \in \mathrm{proj}\, A\}. \end{aligned}$$

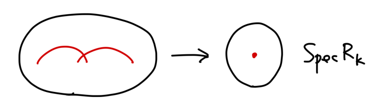
(a) (August) A_{con} is silting-discrete.

(b) (DW) A_{con} represents NC-deformation of $f^{-1}(o)_{\mathrm{red}}$.

(c) (Toda) $\dim A_{\mathrm{con}}$ is related to Gopakumar-Vafa invariant.

Example (c.f. Smith-Wemyss)

If $R_k := \mathbb{C}[\![u,v,x,y]\!]/(uv - xy(x^k + y))$, all resols. look like



- One resol. $f\colon X_1 \to \operatorname{Spec} R_k$ gives

$$A_{\mathrm{con},f} = \bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet$$

  with relations $(ab)^k a = 0 = b(ab)^k$.

- Another resol. $g\colon X_2 \to \operatorname{Spec} R_k$ gives

$$A_{\mathrm{con},g} = \bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \circlearrowleft y$$

  with relations $y^k = ba$ and $ay = 0 = yb$.

Another important aspect of contraction algebras is

(d) For $f\colon X \to \operatorname{Spec} R$, define the null category $\mathcal{C}$ by

$$\mathcal{C} := \{x \in \mathrm{D^b}(\operatorname{coh} X) \mid Rf_*(x) = 0\}.$$

- $\mathcal{C}$ is triangulated subcategory of $\mathrm{D^b}(\operatorname{coh} X)$.
- $\mathcal{C}$ is Hom-finite.
- If X is regular, then $\mathcal{C}$ is 3-CY.

Under VdB's equivalence $\mathrm{D^b}(\operatorname{coh} X) \simeq \mathrm{D^b}(\operatorname{mod} A)$,

$$\mathcal{A} := \mathcal{C} \cap \operatorname{mod} A = \mathcal{C} \cap \operatorname{coh} X = \langle \mathcal{O}_{C_i}(-1) \mid i \rangle_{\mathrm{ex}} \subset \mathcal{C}$$

is called the standard heart, and $\mathcal{A} \simeq \operatorname{mod} A_{\mathrm{con}}$.

- $\mathcal{O}_{C_i}(-1)$ plays important role in both RT and AG.
  - The universal line bundles $\mathcal{O}_{C_i}(-1)$ for each $C_i \simeq \mathbb{P}^1$ gives the full collection of simple objects in $\mathcal{A}$.
  - Each $\mathcal{O}_{C_i}(-1)$ is an example of (fat-)spherical object.

Example

If $f\colon X \to \operatorname{Spec} R$ is the Atiyah flop,

- $R = \mathbb{C}[\![x, y, z, w]\!]/(xy - zw)$,
- $f^{-1}(o) = C \simeq \mathbb{P}^1$,
- $\operatorname{Ext}^i_X(\mathcal{O}_C(-1), \mathcal{O}_C(-1)) = \begin{cases} \mathbb{C} & \text{if } i = 0, 3, \text{ and} \\ 0 & \text{else.} \end{cases}$.

Thus $\mathcal{O}_C(-1)$ is 3-spherical, and

$$T_{\mathcal{O}_C(-1)}(y) := \operatorname{Cone}\left(\operatorname{RHom}_X(\mathcal{O}_C(-1), y) \otimes \mathcal{O}_C(-1) \xrightarrow{\text{ev}} y\right)$$

defines the spherical twist $T_{\mathcal{O}_C(-1)} \in \operatorname{Auteq} \mathrm{D^b}(\operatorname{coh} X)$.

Remark

In general case, by Donovan-Wemyss, the NC-deformation of $\mathcal{O}_C(-1)$ gives the NC twist $T_{\mathcal{O}_C(-1)} \in \operatorname{Auteq} \mathrm{D^b}(\operatorname{coh} X)$

The category $\mathcal{C}$ contains MORE (fat-)spherical objects.

Slogan

Classification of (fat-)sphericals in $\mathcal{C}$ $\leftrightarrow$ structure of $\mathbf{Auteq}\,\mathcal{C}$

(Fat-)spherical objects are also related to

- space of Bridgeland stability conditions, and
- Lagrangian submfd. of the A-side under the mirror symmetry.

Theorem (H-Wemyss)

The realisation functor $\mathbf{D^b}(A_{\mathbf{con}}) \to \mathcal{C}$ of the standard heart

(1) is NEVER an equivalence, but

(2) gives a bijection between brick complexes in $\mathbf{D^b}(A_{\mathbf{con}})$ and bricks (= fat-spherical objects) in $\mathcal{C}$, and

(3) gives a bijection between t-structures.

This gives more context to the main theorem!!

Contraction algebras have more and more aspects.

(e) Donovan-Wemyss conjecture (proved by Muro-Jasso-Keller)

Theorem

Let $\begin{cases} f_1\colon X_1 \to \operatorname{Spec} R_1 \\ f_2\colon X_2 \to \operatorname{Spec} R_2 \end{cases}$ be two 3d flopping conts.
Assume that X_1 and X_2 are regular. Then

$$\mathrm{D}^{\mathrm{b}}(A_{\mathrm{con},f_1}) \simeq \mathrm{D}^{\mathrm{b}}(A_{\mathrm{con},f_2}) \text{ iff } R_1 \simeq R_2.$$

D-equiv. class of contraction algebras classify smooth 3d flops.

(f) Brown-Wemyss conjecture (on f.d. Jacobi algebra, still open)

Proof of Main Theorem

Recall that the main theorem is

Main Theorem (H-Wemyss)

For $x \in \mathrm{D}^{\mathrm{b}}(A)$, the following hold.

(1) $\mathrm{Ext}^{<0}(x, x) = 0$ iff $\exists \mathcal{H}$ length ♡ on $\mathrm{D}^{\mathrm{b}}(A)$ s.t. $x \in \mathcal{H}$.

(2) x is semibrick cpx. iff $\exists x'$ s.t. $x \oplus x' \in \mathrm{smc}\, A$.

(2) follows from (1). The proof of this is routine.

- By (1), $\exists \mathcal{H}$ length ♡ on $\mathrm{D}^{\mathrm{b}}(A)$ s.t. $x \in \mathcal{H}$.
- KY-bij gives $P \in \mathrm{silt}\, A$ s.t. $\mathcal{H} \simeq B := \mathrm{End}_A(P)$.
- Silting-discreteness of A shows B is τ-tilting finite.
- Using Asai's result gives a 2-term smc $x \oplus x' \in \mathrm{smc}\, B$.
- $\mathrm{real}\colon \mathrm{D}^{\mathrm{b}}(B) \to \mathrm{D}^{\mathrm{b}}(A)$ sends the 2-smc $x \oplus x' \in \mathrm{smc}\, B$ to an smc $x \oplus \mathrm{real}(x') \in \mathrm{smc}\, A$.

Proof of (1).

- By KY-bij, $S \in \operatorname{smc} A$ associates $\mathcal{A}_S \subset \mathrm{D^b}(A)$ (bdd ♡).

Notation

For $x \in \mathrm{D^b}(A)$, $S \in \operatorname{smc} A$, and $a \leq b \in \mathbb{Z}$,

$$x \in [a,b]_S :\Leftrightarrow H^i_{\mathcal{A}_S}(x) = 0 \text{ for all } i < a \text{ and } b < i$$

- Let $x \in \mathrm{D^b}(A)$ be a complex with $\operatorname{Ext}^{<0}(x,x) = 0$. Assume for some $S \in \operatorname{smc} A$, $x \in [a,b]_S$.
- Put

$$\Delta_S(x) := \{T \in \operatorname{smc} A \mid S \geq T \geq S[1], x \in [a,b]_T\}.$$

- Since A is silting discrete, $\Delta_S(x)$ is FINITE POSET.

Key Lemma

If $S' \in \Delta_S(x)$ is maximal, then $x \in [a+1,b]_{S'}$.

Repeating this shows

$\exists S'' \in \operatorname{smc} A$ s.t. $x \in [b,b]_{S''} \Leftrightarrow x \in \mathcal{A}_{S''}[-b]$.

(More) geometric counterpart

Important set: $\{T \mid S \geq T \geq S[1]\} = \mathbf{2}$-term smcs w.r.t. S.

(A part of) HomMMP by Wemyss

One can visualise the set

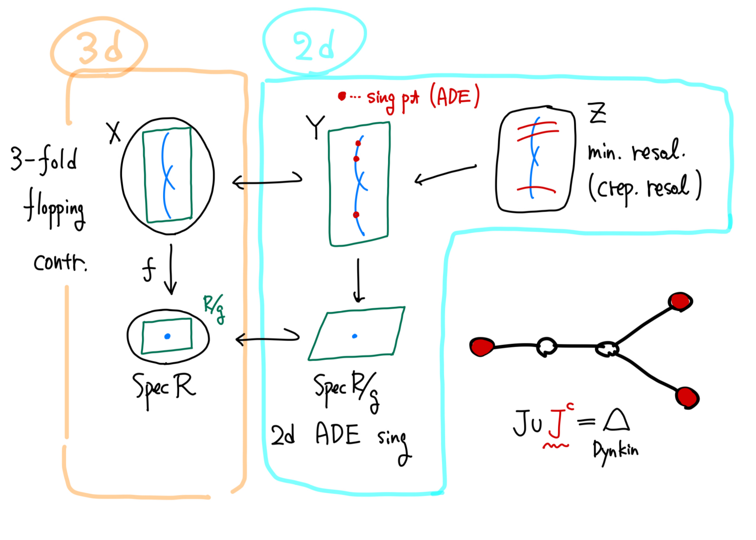
$$\mathbf{2}\text{-smc}\, A_{\mathrm{con}} \simeq \{P \in \operatorname{silt} A_{\mathrm{con}} \mid A_{\mathrm{con}} \geq P \geq A_{\mathrm{con}}[1]\}$$

using Dynkin hyperplane arrangement.

$\mathbf{3}$-fold flops associate marked Dynkin data: general $g \in R$ gives

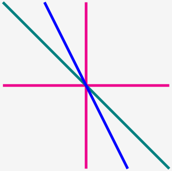
$$\begin{array}{ccccc} X & \longleftarrow & Y & \longleftarrow & Z \\ {\scriptstyle f}\downarrow & \square & \downarrow & & \\ \operatorname{Spec} R & \longleftarrow & \operatorname{Spec} R/g, & & \end{array}$$

- R/g is Kleinian singularity ($=$ surface ADE singularity).
- Y is a partial crepant resolution of R/g.
- Z is the minimal resolution of R/g.


3d
2d
• ··· sing pt (ADE)
X
Y
Z
min. resol.
(crep. resol)
3-fold
flopping
contr.
f
R/g
Spec R
Spec R/g
2d ADE sing
J ∪ J^c = Δ
Dynkin

$\Delta = J \cap J^c$ marked Dynkin data

- $\mathfrak{h} = \bigoplus_{i \in \Delta} \mathbb{R}\,\alpha_i \supset \mathfrak{h}_J := \bigoplus_{i \in J} \mathbb{R}\,\alpha_i$
- $\pi_J \colon \mathfrak{h} \to \mathfrak{h}_J$ natural projection.
- Define the set of positive restricted roots by
$$\{\beta = \pi_J(\alpha) \mid \alpha \in \mathfrak{h} \text{ positive root}, \pi_J(\alpha) \neq 0\}$$
- The associated hyperplane arrangement is
$$\{H_\beta = \beta^\perp \subset \Theta_J := \mathfrak{h}_J^*\}_\beta.$$
- In the previous example, positive restricted roots are
$$\{(1,0), (0,1), (1,1), (2,1), (2,2)\}$$
and the hyperplane arrangement is



For g-vector aspect of this, see Iyama-Wemyss tits cone paper.

3-fold flopping contraction $f\colon X \to \operatorname{Spec} R$ associates

- the contraction algebra A_{con}.
- the hyperplane arrangement $(\Theta_J, \{H_\beta\})$.

HomMMP by Wemyss

There exist natural bijections among

- $\mathbf{2\text{-silt}}\, A_{\mathrm{con}} = \mathbf{2\text{-smc}}\, A_{\mathrm{con}}$.
- Chambers of $(\Theta_J, \{H_\beta\})$.
- (Iterated) flops of X

Flop of X = another 3d flopping contr. $g\colon X' \to \operatorname{Spec} R$

- obtained by modifying some $\mathbb{P}^1$s in $f^{-1}(0)_{\mathrm{red}}$.
- By Bridgeland-Chen, ∃ derived equivalence

$$\mathrm{D^b}(\operatorname{coh} X) \simeq \mathrm{D^b}(\operatorname{coh} X')$$

- Repeating flops give many other models (and all are D-equiv)

Let $f\colon X \to \operatorname{Spec} R$ be a 3d flopping contraction.

Theorem (August)

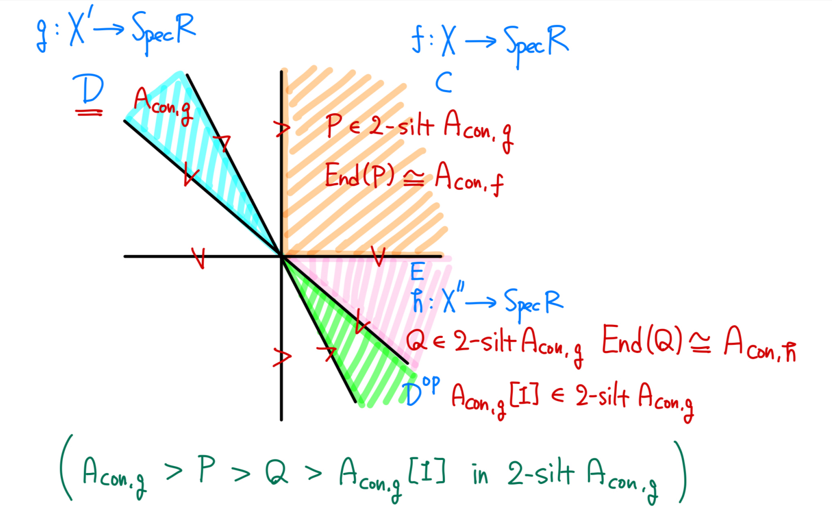
For any ♡ of a bdd t-str. $\mathcal{A} \subset \mathrm{D}^{\mathrm{b}}(A_{\mathrm{con},f})$, there exists another model $g\colon X' \to \operatorname{Spec} R$ such that

- $\mathcal{A} \simeq \operatorname{mod} A_{\mathrm{con},g}$.
- The realisation functor $\mathrm{D}^{\mathrm{b}}(A_{\mathrm{con},g}) \to \mathrm{D}^{\mathrm{b}}(A_{\mathrm{con},f})$ of $\mathcal{A}$ is an equiv.

Thus for $P \in \operatorname{silt} A_{\mathrm{con},f}$, there exist $g\colon X' \to \operatorname{Spec} R$ s.t. $\operatorname{End}(P) \simeq A_{\mathrm{con},g}$ and

$$\begin{aligned}\{Q \mid P \geq Q \geq P[1]\} &\xleftrightarrow{\text{bij}} \text{2-silt } A_{\mathrm{con},g} \\ &\xleftrightarrow{\text{bij}} \text{Iterated flops of } X' \\ &= \text{Iterated flops of } X \\ &\xleftrightarrow{\text{bij}} \text{Chambers of } (\Theta_J, \{H_\beta\}).\end{aligned}$$

$g\colon X' \to \operatorname{Spec} R$ corresp. to $D \subset \Theta_J \setminus \bigcup H_\beta$.
The partial order on $\mathbf{2}\text{-silt}\, A_{\mathrm{con},g}$ can be visualised as:



- By August, for two models
$$g\colon X' \to \operatorname{Spec} R \text{ and } h\colon X'' \to \operatorname{Spec} R,$$
there exists an equivalence
$$\Phi_{g,h}\colon \ \mathrm{D^b}(A_{\mathrm{con},g}) \to \mathrm{D^b}(A_{\mathrm{con},h})$$
defined by $P \in$ 2-silt $A_{\mathrm{con},g}$ with $\operatorname{End}(P) \simeq A_{\mathrm{con},h}$.
- There is a similar equivalence between null categories
$$\Psi_{g,h}\colon \mathcal{C}_g \to \mathcal{C}_h$$
where $\mathcal{C}_f := \{x \in \mathrm{D^b}(\operatorname{coh} X) \mid Rf_*(x) = 0\}$ s.t.
  - The diagram
$$\begin{array}{ccc} \mathrm{D^b}(A_{\mathrm{con},g}) & \xrightarrow{\mathrm{real}} & \mathcal{C}_g \\ \downarrow{\scriptstyle \Phi_{g,h}} & & \downarrow{\scriptstyle \Psi_{g,h}} \\ \mathrm{D^b}(A_{\mathrm{con},h}) & \xrightarrow{\mathrm{real}} & \mathcal{C}_h \end{array}$$
commutes
  - $\Psi_{g,h}$ is a composition of Bridgeland-Chen equivalences.

Main Theorem (geometric version)

Let $f\colon X \to \operatorname{Spec} R$ be a 3d flopping contraction

(1) For $x \in \mathrm{D^b}(A_{\mathrm{con},f})$ (resp. $\mathcal{C}_f$), TFAE

(a) $\operatorname{Ext}^i(x,x) = 0$ for all $i < 0$.

(b) There exists another $f'\colon X' \to \operatorname{Spec} R$ and an equivalence $\Phi\colon \mathrm{D^b}(A_{\mathrm{con},f}) \to \mathrm{D^b}(A_{\mathrm{con},f'})$ (resp. $\Psi\colon \mathcal{C}_f \to \mathcal{C}_{f'}$) s.t.

- $\Phi(x)$ (resp. $\Psi(x)$) $\in \operatorname{mod} A_{\mathrm{con},f'}[n]$ for some $n \in \mathbb{Z}$.
- Φ (resp. Ψ) is a comp. of $\Phi_{g,h}$ (resp. $\Psi_{g,h}$).

(2) If $x \in \mathrm{D^b}(A_{\mathrm{con},f})$ (resp. $\mathcal{C}_f$) satisfies

$$\operatorname{Hom}(x,x) = \mathbb{C} \text{ and } \operatorname{Ext}^i(x,x) = 0 \text{ for all } i < 0,$$

there exists another $f'\colon X' \to \operatorname{Spec} R$ and an equivalence

$$\Phi\colon \mathrm{D^b}(A_{\mathrm{con},f}) \to \mathrm{D^b}(A_{\mathrm{con},f'}) \text{ (resp. } \Psi\colon \mathcal{C}_f \to \mathcal{C}_{f'}) \text{ s.t.}$$

- $\Phi(x)$ (resp. $\Psi(x)$) is a (shifted) simple module.
- Φ (resp. Ψ) is a comp. of $\Phi_{g,h}$ (resp. $\Psi_{g,h}$).

Further remarks

- (2) of the theorem also extends to semibricks.
- Compare the geometric theorem with non-geometric one.

Main Theorem (non-geometric version)

For $x \in \mathbf{D}^{\mathrm{b}}(A)$, the following hold.

(1) $\mathbf{Ext}^{<0}(x,x) = 0$ iff $\exists \mathcal{H}$ length $\heartsuit$ on $\mathbf{D}^{\mathrm{b}}(A)$ s.t. $x \in \mathcal{H}$.

(2) x is semibrick cpx. iff $\exists x'$ s.t. $x \oplus x' \in \mathbf{smc}\, A$.

- A similar technique also classifies all bdd. t-strs. on $\mathcal{C}$.
- As corollaries, theorems yield
  - $\mathbf{Stab}\, \mathcal{C}$ is connected.
  - $\mathbf{Auteq}^{\mathbf{FM}}\, \mathcal{C} \simeq$ the associated pure braid group.
- Geometric theorem also holds for the null category $\mathcal{C}$ of partial crepant resolutions of 2d ADE singularities.

- $\mathbf{2}$d contraction algebras are more complicated than $\mathbf{3}$d.
  - They are (contracted) preprojective algebras, and silting discreteness is still open in many cases.
  - There are less equivalences than $\mathbf{3}$d cases, and no commutative diagram for $\Phi_{g,h}$ and $\Psi_{g,h}$.
- Once the homological mirror for the null category $\mathcal{C}$ is proved, main theorem gives a lot of dynamical and topological corollaries in symplectic geometry.
  - HMS for $R_k := \mathbb{C}[\![u,v,x,y]\!]/(uv - xy(x^k + y))$ is known (Smith-Wemyss), for example.
  - Bricks corresponds to Lagrangian submfds. via the realisation functor and HMS.
  - In Smith-Wemyss, techniques in silting-discrete world (implicitly but actually) contribute to prove results in symplectic geometry!!

Corollary (Smith-Wemyss)

Let $L \subset W_p$ be a closed Lagrangian submfd. with vanishing Maslov class. Then $\pm[L] \in \{(1,0),(0,1),(1,\pm 1)\} \in H^3(W_p;\mathbb{Z}) \simeq \mathbb{Z}\oplus\mathbb{Z}$.