τ -Tilting finiteness of group algebras and p-hyperfocal subgroups

Naoya Hiramae (Kyoto University)
based on a joint work with
Yuta Kozakai (Tokyo University of Science)

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Notation

- k : algebraically closed field
- Λ : finite dimensional k-algebra
- $\operatorname{mod}\Lambda$: the category of right Λ-module of finite dimension
- $\operatorname{proj}\Lambda$: the full subcategory of $\operatorname{mod}\Lambda$ consisting of projectives

For $M \in \operatorname{mod}\Lambda$,

- DM : the k-dual of M
- $|M|$: the number of noniso. indec. summands of M
- τM : the Auslander-Reiten translate of M

§1. τ-Tilting theory and τ-tilting finite algebras

$\boxed{\text{Classical tilting theory}}$

tilting modules $\rightsquigarrow$ derived equiv. algebras

$\boxed{\tau\text{-tilting theory}}$

$$\begin{array}{ccc} \text{support } \tau\text{-tilting modules} & \xleftrightarrow{1:1} & \text{2-term silting complexes} \\ & & \cup\!\shortmid \;\; \| \\ & & \text{2-term tilting complexes} \rightsquigarrow \text{derived equiv. algebras} \end{array}$$

over symmetric algebras

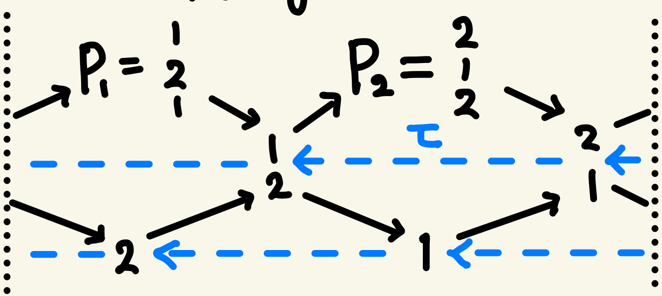
Def. • $M \in \operatorname{mod}\Lambda$ is a support τ-tilting module

$$:\Longleftrightarrow \begin{cases} M : \tau\text{-rigid}, \text{ i.e. } \operatorname{Hom}_\Lambda(M, \tau M) = 0, \\ \exists P \in \operatorname{proj}\Lambda,\ \operatorname{Hom}_\Lambda(P, M) = 0 \text{ and } |P| + |M| = |\Lambda|. \end{cases}$$

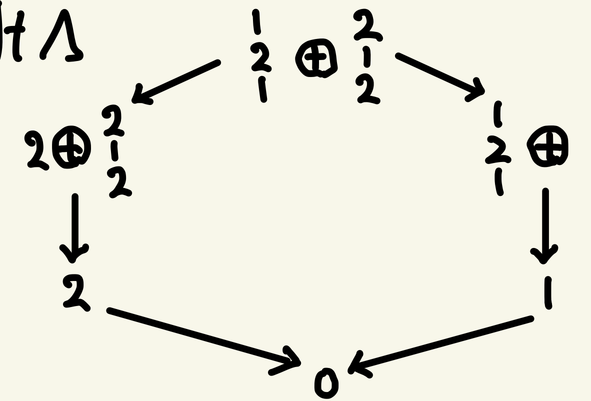
• $s\tau\text{-tilt}\,\Lambda := \{\text{basic support } \tau\text{-tilting modules over } \Lambda\}/\text{iso}$

Example $\operatorname{char} k = 3$, $\Lambda := k\mathfrak{S}_3 \cong k[1 \underset{b}{\overset{a}{\rightleftarrows}} 2]/(aba, bab)$

AR-quiver



$s\tau$-tilt Λ



<u>Def.</u>

- $M \in \operatorname{mod}\Lambda$ is a brick $:\Leftrightarrow \operatorname{End}_\Lambda(M) \cong k$
- $\operatorname{brick}\Lambda := \{\text{bricks over } \Lambda\}/\text{iso}$

<u>Def.</u> Λ is τ-tilting finite $:\Leftrightarrow \#\, s\tau\text{-tilt}\,\Lambda < \infty$

<u>Prop.</u> [Demonet-Iyama-Jasso] TFAE.

(a) Λ is τ-tilting finite.

(b) $\#\operatorname{brick}\Lambda < \infty$.

(c) Every torsion class in $\operatorname{mod}\Lambda$ is functorially finite.

Remark

- $\begin{cases} \Lambda \overset{\exists}{\twoheadrightarrow} \Gamma : \text{surj. alg. hom.} \\ \Gamma : \tau\text{-tilting infinite} \end{cases} \Longrightarrow \Lambda : \tau\text{-tilting infinite}.$

- $\Lambda := kQ$ (Q : acyclic quiver) : τ-tilting finite $\overset{\text{iff}}{\Longleftrightarrow}$ Q : Dynkin.

- We can show τ-tilt. inf. by the shape of quivers in some cases.

e.g.)
$$\Lambda := k\left[\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[d, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \\ \bullet \arrow[u, shift left] \arrow[r, shift left] & \bullet \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}\right] \Big/ \sim \;\twoheadrightarrow\; k\left[\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \bullet \arrow[l] \arrow[u] \end{tikzcd}\right] : \tau\text{-tilting infinite}$$

arbitrary (admissible) relation

$\therefore\ \Lambda : \tau$-tilting infinite

Question Does derived equiv. preserve τ-tilting finiteness?

↳ It's false in general,

but no counterexamples are found over symmetric alg.

Moreover, it's shown to be true for the following cases:

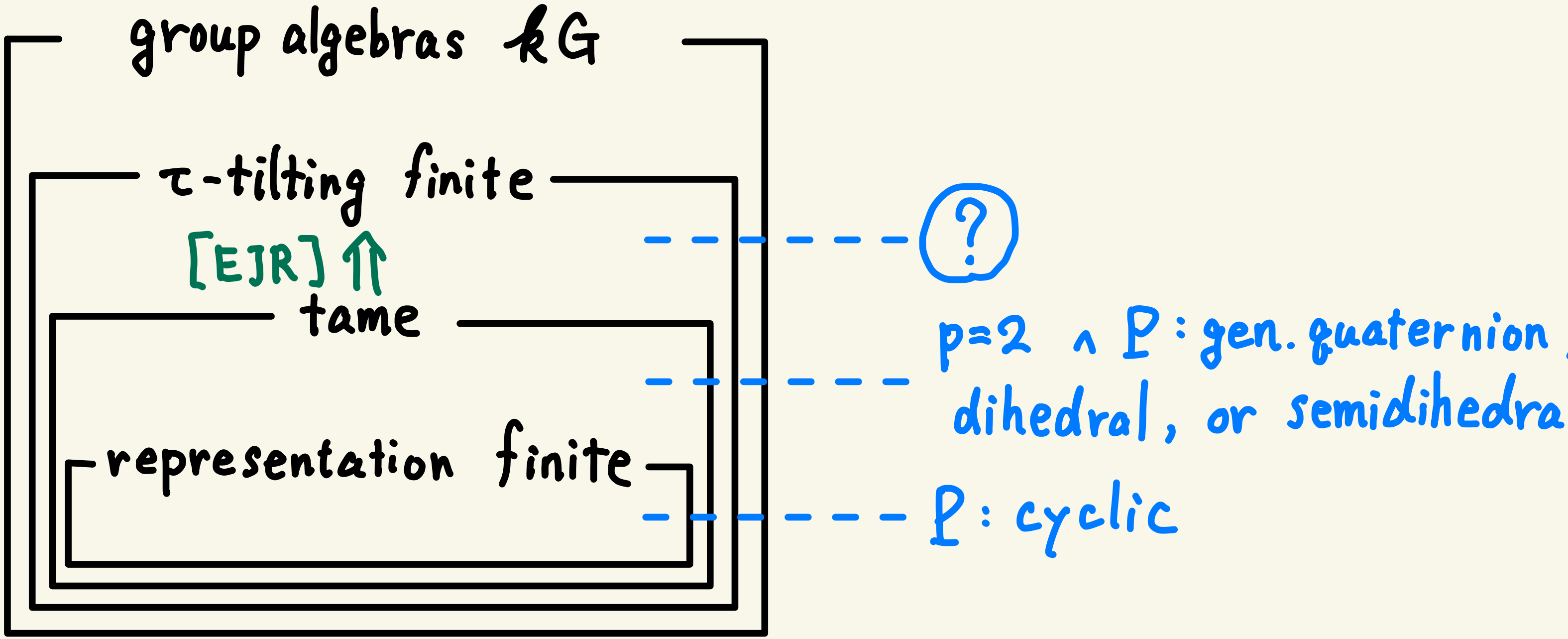
- [Miyamoto-Wang] symmetric alg. of polynomial growth
- [Adachi-Aihara-Chan] Brauer graph alg. ← symmetric

If τ-tilting finiteness of symmetric alg. is invariant under derived equivalences, then τ-tilting finiteness implies tilting connectivity over symmetric alg. [Aihara-Mizuno]

$\forall T_1, T_2$: tilting cpx., T_1 is obtained from T_2 by irreducible mutations.

§2. τ-Tilting finite group algebras

$p := \mathrm{char}\, k > 0$, G: finite group, P: Sylow p-subgrp. of G



<u>Question</u> What controls τ-tilting finiteness of kG ?

Remark τ-Tilting finiteness of a group algebra kG is NOT determined by its Sylow p-subgroup P.

e.g.) $k[C_p \times C_p]$: τ-tilting finite.

$k[(C_p \times C_p) \underline{\rtimes} C_2]$: τ-tilting infinite for $\forall p \neq 2$.

sending to the inverse

Def. We call $P \cap \underline{O^p(G)}$ a p-hyperfocal subgroup of G.

the smallest normal subgrp. of G s.t. its quotient is a p-group

	G	P	$O^p(G)$	$P \cap O^p(G)$
e.g.)	$C_p \times C_p$	$C_p \times C_p$	1	1
	$(C_p \times C_p) \rtimes C_2$	$\parallel$ $C_p \times C_p$	$(C_p \times C_p) \rtimes C_2$	$\nparallel$ $C_p \times C_p$

$R := P \cap O^p(G)$: a p-hyperfocal subgroup of G

Prop. [Koshio-Kozakai]
$kO^p(G)$: τ-tilting finite $\Rightarrow$ kG : τ-tilting finite.

Cor. kG is τ-tilting finite if one of the following holds:
(a) R is cyclic.
(b) $p=2$ and R is dih., semidih., or gen. quat.

$\because$ Since R is a Sylow p-subgrp. of $O^p(G)$,
(a) or (b) $\Rightarrow$ $kO^p(G)$: tame $\Rightarrow$ $kO^p(G)$: τ-tilt. fin. $\Rightarrow$ kG : τ-tilt. fin.

Our conjecture The converse of Cor. holds.

Thm. [H-Kozakai] P : abelian p-group, H: abelian p'-group acting on P, $G := P \rtimes H$.
Then kG is τ-tilt. fin. iff one of the following holds:

(a) $p=2$ and R is trivial or $C_2 \times C_2$.

(b) $p \geq 3$ and R is cyclic.

(p'-group := group whose order is coprime to p.)

Remark In the above setting, $R = [P, H]$.
R can not be nontrivial cyclic if $p=2$.

§3. Sketch of proof of Thm.

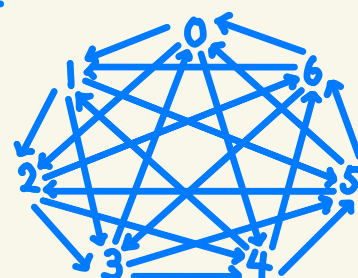
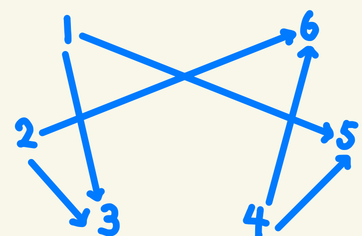
$G := P \rtimes H$ $\left(\begin{array}{l} P : \text{abelian } p\text{-grp.} \\ H : \text{abelian } p'\text{-grp.} \end{array} \right)$

We know the quiver and relations for kG.

Then we can take τ-tilt. inf. quotient algebras of kG

such as $k[\cdot \rightrightarrows \cdot]$, $k\left[\begin{tikzcd}[sep=small] & \cdot \arrow[dr] & \\ \cdot \arrow[ur] \arrow[rr] & & \cdot \end{tikzcd}\right]$, $k\left[\begin{tikzcd}[sep=small] \cdot \arrow[r] \arrow[d] & \cdot \\ \cdot & \cdot \arrow[u] \arrow[l] \end{tikzcd}\right]$, $\cdots$.

e.g.) • $k[(C_p \times C_p) \rtimes C_2] \cong k\left[\cdot \substack{\rightrightarrows \\ \leftleftarrows} \cdot\right]/\sim \ \twoheadrightarrow k[\cdot \rightrightarrows \cdot]$ $(p \geq 3)$

• $k[(C_2)^3 \rtimes C_7] \cong k\Big[$  $\Big]/\sim \ \twoheadrightarrow k\Big[$  $\Big]$.

$(p = 2)$

- $p = 2$, $G := \underbrace{(C_{2^\ell})^2}_{\langle a \rangle \times \langle b \rangle} \rtimes \underbrace{C_3}_{\langle c \rangle}$

  $c : a \mapsto b \mapsto a^{-1}b^{-1}$

$$
kG \cong \frac{k\left[\begin{tikzcd} & 2 \arrow[dr, bend left, "\alpha"] \arrow[dl, "\beta"'] & \\ 1 \arrow[ur, bend left, "\alpha"] \arrow[rr, "\beta"'] & & 3 \arrow[ll, bend left, "\alpha"] \arrow[ul, "\beta"'] \end{tikzcd}\right]}{\left(\begin{matrix} \alpha\beta - \beta\alpha \\ \alpha^{2^\ell}, \beta^{2^\ell} \end{matrix}\right)} \twoheadrightarrow \begin{cases} k\left[\begin{tikzcd} & 2 \arrow[dr] & \\ 1 \arrow[ur] \arrow[rr] & & 3 \end{tikzcd}\right] & (\ell \geq 2) : \tau\text{-tilt. inf.} \\ \dfrac{k\left[\begin{tikzcd} & 2 \arrow[dr, "\alpha_2"] & \\ 1 \arrow[ur, "\alpha_1"] \arrow[rr] & & 3 \end{tikzcd}\right]}{(\alpha_2\alpha_1)} & (\ell = 1) : \tau\text{-tilt. fin.} \end{cases}
$$

Remark The above method does not work in general.

e.g.) $p \geq 5$, $G := (C_p)^3 \underline{\rtimes} \mathfrak{S}_3$

permuting entries

$$kG \underset{\text{Morita}}{\sim} k\left[\circlearrowleft 1 \rightleftarrows \overset{\circlearrowleft}{\underset{\circlearrowleft}{2}} \rightleftarrows 3 \circlearrowright\right] / \sim$$

We cannot construct any τ-tilt. inf. path alg. as quotients of kG.

But kG is τ-tilting infinite by the following:

> **Thm.** [H] Let H be a subgrp. of $\mathfrak{S}_n$. If $p^\ell \geq n$ and $|kH| \geq \min\{p, 3\}$, then $k[(C_{p^\ell})^n \rtimes H]$ is τ-tilting infinite.

§4. Why we consider $k[P \rtimes H]$ (P: p-grp., H: p'-grp.)

G : finite group, P : Sylow p-subgrp. of G

$B_0(kG)$: the principal block of kG

block of kG := indec. summand of kG as a kG-bimodule.

the principal block of kG := the unique block B of kG s.t. $k_G B \neq 0$. (k_G : trivial kG-module)

Broué's Abelian Defect Conjecture If P is abelian, then $B_0(kG)$ and $B_0(kN_G(P))$ are derived equivalent.

Remark By the Schur-Zassenhaus theorem,

${}^\exists H$: p'-subgrp. of $N_G(P)$ s.t. $N_G(P) = P \rtimes H$.

mentioned in §1

Conjecture τ-Tilting finiteness of symmetric alg. is invariant under derived equivalences.

If we assume that $\begin{cases} \cdot \text{ the above conjectures hold, and} \\ \cdot\ P \text{ is abelian,} \end{cases}$ too strong assumption...

then our conjecture can be reduced to the case $G = P \rtimes H$.

Our conj. If P is abelian, then kG is τ-tilting finite iff a p-hyperfocal subgrp. is cyclic or $C_2 \times C_2$.