Quantum projective planes and Beilinson algebras of 3-dimensional quantum polynomial algebras for Type S'

Ayako Itaba

Tokyo University of Science, Institute of Arts and Sciences

September 7th, 2023

The 55th Symposium on Ring Theory and Representation Theory @Osaka Metropolitan University

Quantum polynomial algebras

- k: an algebraically closed field with $\mathrm{char}\, k = 0$,
- A: a connected graded k-algebra fin. gen. in degree 1.

Definition 1.1 (Artin-Schelter, 1987)

A right noetherian graded algebra A is called a *d-dimensional quantum polynomial algebra* (*d-dim qpa*) if

1. $\mathrm{gldim}\, A = d < \infty$,
2. $\mathrm{Ext}^i_A(k, A) \cong \begin{cases} k & \text{if } i = d, \\ 0 & \text{if } i \neq d, \end{cases}$ (*Gorenstein condition*)
3. $H_A(t) := \sum_{i=0}^{\infty} (\dim_k A_i) t^i = (1-t)^{-d}$ (*Hilbert series*).

- A right noetherian graded algebra A is called a d-dimensional AS-regular algebra if the above conditions (i) and (ii) hold.
- A: 3-dim qpa $\Longleftrightarrow$ A: 3-dim. quadratic AS-regular alg.

Quantum projective spaces (quantum $\mathbb{P}^{d-1}$)

- A: a right noeth. graded algebra.
- grmod$A$: the cat. of finitely generated graded right A-modules,
- tors A: the full subcat. of grmod A consisting of fin. dim. modules over k.

Definition 1.2 (Artin-Zhang, 1994)

1. *The noncommutative projective scheme associated to A* is defined by $\text{Proj}_{\text{nc}}A = (\text{tails}\, A, \pi A)$ where
   - $\text{tails}\, A := \text{grmod}\, A/\text{tors}\, A$ is the quot. cat.,
   - $\pi : \text{grmod}\, A \to \text{tails}\, A$ is the quot. func., $A \in \text{grmod}\, A$ is regular.
2. A: d-dim qpa $\Longrightarrow \text{Proj}_{\text{nc}}A$ is called *a quantum $\mathbb{P}^{d-1}$*.
   - $d = 3 \Longrightarrow \text{Proj}_{\text{nc}}A$ is called *a quantum projective plane*.

Remark 1.3

- A: commutative $\Longrightarrow \text{Proj}_{\text{nc}}\, A \cong (\text{mod}\, X, \mathcal{O}_X)$, $X = \text{Proj}\, A$.
- A: 2-dim qpa $\Longrightarrow \text{Proj}_{\text{nc}}\, A \cong (\text{coh}\, \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$.

Characterization when 3-dim qpa is finite over its center

A geometric pair (E, σ): $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$.

Theorem 2.1 (ATV, 1991)

$A = \mathcal{A}(E, \sigma)$: 3-dim qpa. Then

$$|\sigma| < \infty \iff A \text{ is finite over its center.}$$

- To prove Theorem 2.1, "fat points of a quantum projective plane $\mathrm{Proj_{nc}}\, A$" plays an essential role.
- By [Artin, 1992], if A is finite over its center and $E \neq \mathbb{P}^2$, then $\mathrm{Proj_{nc}}\, A$ has a fat point, however, the converse is not true.

Definition 2.2

Let A be a graded algebra.

1. *A point of* $\mathrm{Proj_{nc}} A$ is an isom. class of a simple obj. of the form $\pi M \in \mathrm{tails}\, A$ where $M \in \mathrm{grmod}\, A$ such that $\lim_{i \to \infty} \dim_k M_i < \infty$.
2. A point πM is called *fat* if $\lim_{i \to \infty} \dim_k M_i > 1$ (in this case, M is called *a fat point module over A*).

Norm $\|\sigma\|$

- To check the existence of a fat point, the following was introduced.

Definition 2.3 (Mori, 2015)

For a geometric pair (E, σ) where $E \subset \mathbb{P}^{n-1}$ and $\sigma \in \mathrm{Aut}_k E$,

$$\mathrm{Aut}_k(\mathbb{P}^{n-1}, E) := \{\phi|_E \in \mathrm{Aut}_k E \mid \phi \in \mathrm{Aut}_k \mathbb{P}^{n-1}\},$$

and $\|\sigma\| := \inf\{i \in \mathbb{N}^+ \mid \sigma^i \in \mathrm{Aut}_k(\mathbb{P}^{n-1}, E)\}$, which is called *the norm of* σ.

- For a geometric pair (E, σ), $\|\sigma\| \leq |\sigma|$ holds.

Lemma 2.4 (Mori, 2015), (Artin, 1992)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa. Then the following hold:

1. $\|\sigma\| = 1 \iff E = \mathbb{P}^2$.
2. $1 < \|\sigma\| < \infty \iff \mathrm{Proj}_{\mathrm{nc}} A$ *has a fat point.*

Properties of $|\sigma|$ and $\|\sigma\|$

- For a d-dim qpa, the following hold in general:

Lemma 2.5 (Mori-Ueyama, 2013), (Mori, 2015)

Let A and A' be d-dim qpa "satisfying the condition $(\mathrm{G1})$, *where* $\mathcal{P}(A) = (E, \sigma)$ *and* $\mathcal{P}(A') = (E', \sigma')$", *respectively. Then the following hold:*

1. $A \cong A' \implies E \cong E',\ |\sigma| = |\sigma'|$.
2. $\operatorname{grmod} A \cong \operatorname{grmod} A' \implies E \cong E',\ \|\sigma\| = \|\sigma'\|$.
   - *In particular, when $d = 3$,*

$$\mathrm{Proj_{nc}} A \cong \mathrm{Proj_{nc}} A' \implies E \cong E',\ \|\sigma\| = \|\sigma'\|.$$

- ([Abdelgadir-Okawa-Ueda, 2014]) Let A and A' be 3-dim qpa. Then $\operatorname{grmod} A \cong \operatorname{grmod} A' \iff \mathrm{Proj_{nc}} A \cong \mathrm{Proj_{nc}} A'$.

Remark

Lemma 2.5 (2) tells us that, for a 3-dim qpa $A = \mathcal{A}(E, \sigma)$, the norm $\|\sigma\|$ of σ is a categorical invariant in $\mathrm{Proj_{nc}} A$.

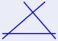
$\mathsf{Proj}_{\mathrm{nc}} A$ is finite over its center

Definition 2.6 ((Mori, 2015), (I.-Mori, 2023))

Let A be a d-dim qpa. We say that $\mathsf{Proj}_{\mathrm{nc}} A$ is *finite over its center* if there exists a d-dim qpa A' finite over its center such that

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \ (\mathsf{Proj}_{\mathrm{nc}} A \cong \mathsf{Proj}_{\mathrm{nc}} A').$$

Theorem 2.7 (Mori, 2015)

$A = \mathcal{A}(E, \sigma)$: a 3-dim qpa where E is a triangle  *in $\mathbb{P}^2$, $\sigma \in \mathrm{Aut}_k E$. ($A$ is called a Type S algebra.) Then*

$$\|\sigma\| < \infty \iff \mathsf{Proj}_{\mathrm{nc}} A \text{ is finite over its center.}$$

Characterization when $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.

Theorem 2.8 (I.-Mori, 2023)

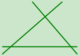
If $A = \mathcal{A}(E, \sigma)$ is a 3-dim Calabi-Yau quantum polynomial algebra, then $||\sigma|| = |\sigma^3|$, so the following are equivalent:

1. $|\sigma| < \infty$.
2. $||\sigma|| < \infty$.
3. *A is finite over its center.*
4. *$\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.*

Theorem 2.9 (I.-Mori, 2023)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa such that $E \neq \mathbb{P}^2$, and $\nu \in \mathrm{Aut} A$ the Nakayama auto. of A. Then the following are equivalent:

1. $|\nu^* \sigma^3| < \infty$.
2. $\|\sigma\| < \infty$.
3. *$\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.*
4. *$\mathrm{Proj}_{\mathrm{nc}} A$ has a fat piont.*

Example 1 (Type S, $E =$  triangle in $\mathbb{P}^2$)

$A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$: 3-dim qpa,
where $\alpha, \beta, \gamma \in k\backslash\{0\}$, $E = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(z) \subset \mathbb{P}^2$,

$$\begin{cases} \sigma(0,b,c) = (0,b,\alpha c), \\ \sigma(a,0,c) = (\beta a,0,c), \\ \sigma(a,b,0) = (a,\gamma b,0), \end{cases} \quad \nu^* = \begin{pmatrix} \gamma/\beta & 0 & 0 \\ 0 & \alpha/\gamma & 0 \\ 0 & 0 & \beta/\alpha \end{pmatrix},$$

$$\begin{cases} \nu^*\sigma^3(0,b,c) = (0,b,\alpha\beta\gamma c), \\ \nu^*\sigma^3(a,0,c) = (\alpha\beta\gamma a,0,c), \\ \nu^*\sigma^3(a,b,0) = (a,\alpha\beta\gamma b,0). \end{cases}$$

1. $|\sigma| = \mathrm{lcm}(|\alpha|, |\beta|, |\gamma|) < \infty$ $\overset{\text{Thm 2.1 by ATV}}{\Longleftrightarrow}$ A is finite over its center.
2. $\|\sigma\| = |\nu^*\sigma^3| = |\alpha\beta\gamma| < \infty$ $\overset{\text{Thm 2.9 by I.-Mori}}{\Longleftrightarrow}$ $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center $\overset{\text{Thm 2.9 by I.-Mori}}{\Longleftrightarrow}$ $\mathrm{Proj}_{\mathrm{nc}} A$ has a fat piont.

Beilinson algebras and Minamoto-Mori correspondence

In [Minamoto-Mori, 2011], for a d-dim qpa A, *the Beilinson algebra ∇A of A* is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{d-1} \\ 0 & A_0 & \cdots & A_{d-2} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

Theorem 2.10 (Minamoto-Mori, 2011)

If A is a d-dim qpa A and the Beilinson algebra ∇A. Then

- *∇A is extremely Fano of global dimension of $d-1$, and*
- *there exists an equivalence of tri. cat.*

$$\mathrm{D^b}(\mathrm{tails}A) \cong \mathrm{D^b}(\mathrm{mod}\nabla A).$$

- The Beilinson algebra is a typical example of $(d-1)$-representation infinite algebra in the sense of [Herschend-Iyama-Oppermann, 2014] ([Minamoto-Mori, 2011]).

Remark

Remark

(1) If A is a 2-dimensional quantum polynomial algebra, then

$$\nabla A \cong \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix} \cong k(\bullet \rightrightarrows \bullet),$$

(∇A is isomorphic to a 2-Kronecker algebra) so ∇A is a finite dimensional hereditary algebra of tame representation type. It is known that the isomorphism classes of simple regular modules over ∇A are parameterized by $\mathbb{P}^1$ (cf. [Mori, 2015]).

(2) For a 3-dim qpa A, ∇A is a finite-dimensional algebra.

$$\nabla A \cong k \left(\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bullet \right) \Big/ \text{(the same relations of } A\text{)}.$$

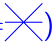
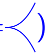
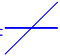
Applications

- We apply our results to Representation theory of finite dimensional algebras.

Corollary 2.11 (I.-Mori, 2023)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dim qpa with the Nakayama auto. $\nu \in \mathrm{Aut}\, A$. Then the following are equivalent:

1. $|\nu^* \sigma^3| (= \|\sigma\|) = 1$ *or* ∞.
2. $\mathrm{Proj}_{\mathrm{nc}} A$ *has no fat point.*
3. *The isomorphism classes of simple 2-regular modules over ∇A are parameterized by the set of closed points of $E \subset \mathbb{P}^2$.*

In particular, if A is of Type P ($E = \mathbb{P}^2$), T ($E=$ ), T' ($E=$ ), CC ($E=$ ), TL ($E=$) or WL ($E=$ ), then A satisfies all of the above conditions.

Example 2 (Type CC, $E =$ )

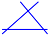
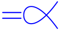
$A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(f_1, f_2, f_3)$: 3-dim qpa,

$$\begin{cases} f_1 = yz - zy + y^2 + 3x^2 \\ f_2 = zx - xz + yx + xy - yz - zy \\ f_3 = xy - yx - y^2 \end{cases}$$

$$\sigma(a, b, c) = \left(a - b, b, -3\frac{a^2}{b} + 3a - b + c\right), \ \nu^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\forall i \geq 1,\ \ \sigma^i(a, b, c) = (a - ib, b, -3i\frac{a^2}{b} + 3i^2 a - i^3 b + c)$, $\sigma^i \notin \mathrm{Aut}(\mathbb{P}^2, E)$.
$||\sigma|| = \infty = |\sigma^3|$.

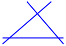
1. By $|\sigma| = \infty$ and Theorem 2.1 (by ATV), A is not finite over its center.
2. By $\|\sigma\| = |\nu^*\sigma^3| (= |\sigma^3|) = \infty$ and Corollary 2.11 (by I.-Mori), $\mathrm{Proj}_{\mathrm{nc}} A$ has not a fat piont, and the isomorphism classes of simple 2-regular modules over ∇A are parameterized by the set of closed points of $E \subset \mathbb{P}^2$.

- $A = \mathcal{A}(E, \sigma)$: 3-dim qpa, $|\nu^* \sigma^3| (= \|\sigma\|) < \infty$ (??).
- (Type S (E= ), S' (E= ), NC (E= ), EC (E= ))

Conjecture 2.12 (I.-Mori, 2023)

For a 3-dimensional quantum polynomial algebra A, we expect that the following are equivalent:

1. $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.
2. ∇A is 2-representation tame in the sense of [Herschend-Iyama-Oppermann, 2014].
3. The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.

- Note that these equivalences are shown for Type S in [Mori 2015]. (Type S: E is a triangle  in $\mathbb{P}^2$.)
- Do these equivalences in Conjecture 2.12 hold for Type S' in particular? (Type S' : E is  in $\mathbb{P}^2$.)

Centers of Calabi-Yau Type S' algebras

Proposition 1 (I., 2023)

Let $A = \mathcal{A}(E, \sigma) = k\langle x, y, z\rangle/(g_1, g_2, g_3)$ be a 3-dimansional Calabi-Yau quantum polynomial algebra of Type S', where

$$\begin{cases} g_1 = yz - \alpha zy + x^2, \\ g_2 = zx - \alpha xz, \\ g_3 = xy - \alpha yx \quad (\alpha^3 \neq 0, 1). \end{cases}$$

Then $g := xyz + (1 - \alpha^3)^{-1}x^3 \in Z(A)_3$.

1. If A is finite over its center $Z(A)$ (that is, $|\alpha| < \infty$), then $Z(A) = k[x^{|\alpha|}, y^{|\alpha|}, z^{|\alpha|}, g]$.
2. If A is not finite over its center $Z(A)$ (that is, $|\alpha| = \infty$), then $Z(A) = k[g]$.

- ([I.-Matsuno, 2022]) $\forall$ 3-dim qpa A, $\exists$ 3-dim Calabi-Yau qpa A' such that $\operatorname{grmod} A \cong \operatorname{grmod} A'$ so that $\operatorname{Proj}_{\mathrm{nc}} A \cong \operatorname{Proj}_{\mathrm{nc}} A'$.

Result for Type S'

Theorem 3.1 (Mori, 2015)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra.
If the Beilinson algebra ∇A of A is not 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $E \subsetneq \mathbb{P}^2$.

Theorem 1 (I., 2023)

Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quantum polynomial algebra of Type S'.
If the Beilinson algebra ∇A of A is 2-representation tame, then the isomorphism classes of simple 2-regular modules over ∇A are parametrized by the set of points of $\mathbb{P}^2$.

Conjecture 2.12 holds for Type S'

Theorem 2 (I., 2023)

For a 3-dimensional quantum polynomial algebra A of Type S', the following are equivalent:

1. $\mathrm{Proj}_{\mathrm{nc}} A$ is finite over its center.
2. ∇A of A is 2-representation tame in the sense of [Herschend-Iyama-Oppermann, 2014].
3. The isomorphism classes of simple 2-regular modules over ∇A are parameterized by $\mathbb{P}^2$.

Thank you for your attention!

- A. Itaba and I. Mori, *Quantum projective planes finite over their centers*, Can. Math. Bull. Vol. **66** (2023), Issue 1, 53–67.
- A. Itaba, Quantum projective planes and Beilinson algebras of 3-dimensional quantum polynomial algebras for Type S', submitted, arXiv:2304.02242.