Auslander-Reiten's Cohen-Macaulay algebras and Contracted preprojective algebras

(jt with Chan, Marczinzik)

[AR 1991] Cohen-Macaulay and Gorenstein Artin algebras

Introduce non-comm. version of $\longrightarrow$ almost forgotten

commutative : regular $\Longrightarrow$ Gorenstein $\Longrightarrow$ Cohen-Macaulay rings

$\mathrm{gl.dim}\,A < \infty$ $\quad$ $\mathrm{inj.dim}\,A < \infty$ $\quad$ $\mathrm{depth}\,A = \mathrm{Krull\ dim}\,A$

A : Noeth. ring $\quad$ $P^{<\infty}(A) := \{ X \in \mathrm{mod}A \mid \mathrm{proj.dim}_A X < \infty \}$

$I^{<\infty}(A) \quad \{ \quad \mathrm{inj.dim} \quad \}$

<u>Def</u> A-bimodule W is called a dualizing module of A

$\overset{\text{def}}{\Longleftrightarrow}$ $W \otimes_A - : P^{<\infty}(A) \simeq I^{<\infty}(A) : \mathrm{Hom}_A(W, -)$

In this case, call A Cohen-Macaulay (CM)

<u>Ex</u> ① [Sharp] $\forall$ CM local ring A ($\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{K.dim}\,A = \mathrm{depth}\,A$) with canonical module ω

is CM in the sense above, ω is a dualizing module

② Iwanaga-Gorenstein rings A ($\Longleftrightarrow$ $P^{<\infty}(A) = I^{<\infty}(A)$) are precisely

CM rings with dualizing module A

<u>In the rest, A : fin. dim alg. / field k</u>

<u>Def</u> $U \in \mathrm{mod}\,A$: cotilting $\Longleftrightarrow$ ① $\mathrm{inj.dim}_A U < \infty$ ② $\mathrm{Ext}^i_A(U, U) = 0 \quad \forall i \geq 1$

③ $\exists$ exact $0 \longrightarrow \underbrace{U_n \to \cdots \to U_0}_{\mathrm{add}\,U} \longrightarrow DA \longrightarrow 0$

$\mathrm{cotilt}A := \{ \text{cotilting } A\text{-modules} \} / \underset{\mathrm{add}}{\sim}$: poset

$\ni U, V \qquad U \geq V \overset{\text{def}}{\Longleftrightarrow} \mathrm{Ext}^i_A(U, V) = 0 \quad \forall i \geq 1$

<u>Rem</u> $\mathrm{cotilt}\,A$ has a minimal elem. DA, does not necessarily has a max. elem

<u>Prop</u> [AR] A-bimodule W is dualizing $\Longleftrightarrow$

- ${}_AW$ is a max. elem. of $\mathrm{cotilt}A$, $A \xrightarrow{\sim} \mathrm{End}({}_AW)^{op}$ $\quad a \mapsto (w \mapsto wa)$
- W_A ———————— $\mathrm{cotilt}\,A^{op}$, $A \xrightarrow{\sim} \mathrm{End}(W_A)$ $\quad a \mapsto (w \mapsto aw)$

In this case, $\mathrm{fin.dim}\,A = \mathrm{inj.dim}\,W_A = \mathrm{inj.dim}\,{}_AW = \mathrm{fin.dim}\,A^{op}$

$\mathrm{fin.dim}\,A = \sup\{ \mathrm{proj.dim}_A X \mid X \in P^{<\infty}(A) \}$

$\mathrm{fin.dim}\,A^{op} = \sup\{ \mathrm{inj.dim}_A X \mid X \in I^{<\infty}(A) \}$

The proof is based on Auslander-Reiten correspondence

(König-Yang corresp. の原型の1つ)

$U \in \operatorname{cotilt} A \overset{1-1}{\longleftrightarrow} \{ \mathcal{X} \mid \text{contrav. fin. resol. subcat. of } \operatorname{mod} A \text{ s.t. } \mathcal{X} \supset \overset{\exists}{\Omega^n}(\operatorname{mod} A) \} \ni {}^{\perp}U$

$\overset{1-1}{\longleftrightarrow} \{ \mathcal{Y} \mid \text{cov fin. coresol. subcat. of } \operatorname{mod} A \text{ s.t. } \mathcal{Y} \subset I^{<\infty}(A) \} \ni ({}^{\perp}U)^{\perp} = \widehat{\operatorname{add} U}$

<u>Ex</u> ① $\text{fin.dim } A = 0 \iff \operatorname{soc} A_A :$ sincere
$\text{fin.dim } A^{op} = 0 \iff \operatorname{soc} {}_AA :$ sincere
$\Big\} \iff A :$ CM with dualizing mod DA

② $A, B :$ CM $\Rightarrow A \otimes_k B :$ CM

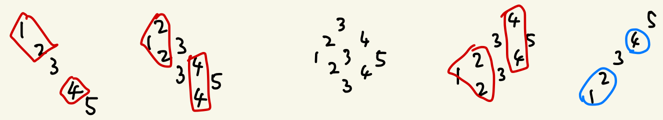
No other CM algebras seem to be known

<u>Def</u> $\Delta :$ Dynkin diagram $\quad \Pi :$ preprojective algebra of type Δ

$J \subset \Delta_0$, $\quad e := \sum_{i \in J} e_i \qquad \Pi(\Delta, J) := e\Pi e \;:$ contracted preprojective algebra

<u>Ex</u> $\Delta = A_5 \qquad \Pi = k\Big[1 \underset{a^*}{\overset{a}{\rightleftarrows}} 2 \rightleftarrows 3 \rightleftarrows 4 \rightleftarrows 5 \Big] \Big/ \Big(\sum_a (aa^* - a^*a) \Big)$

$J = \{1, 2, 4\}$



$P_1 \qquad P_2 = I_4 \qquad\qquad P_4 = I_2 \qquad I_1$

<u>Thm</u> [CIM] $\quad \Delta :$ Dynkin, $J \subset \Delta_0$. $\quad A := \Pi(\Delta, J)$

① A is a CM alg. of $\text{fin.dim } A = \begin{cases} 0 & \text{if } J_m = \phi \\ 2 & \phantom{\text{if }} \neq \phi \end{cases}$

② A has a dualizing module $\mu^+_{J_m} \circ \mu^+_{J_m}(DA) \qquad (= DA \text{ if } \text{fin.dim} A = 0)$

$\mu^+_{J_m} :$ simultaneous mutation at J_m

$\iota : \Delta_0 \simeq \Delta_0 :$ Nakayama perm. of Π

$J_f := \Big\{ i \in J \;\Big|\; \overset{\exists}{\text{path}} \text{ in } \underset{\ell \geq 0}{\Delta} \quad i - \underbrace{i_1 - \cdots - i_{\ell-1}}_{\notin J} - i_\ell \in \iota(J) \Big\} :$ frozen vertices

$J_m := J \setminus J_f \;:$ mutable vertices

<u>Ex</u> $\underset{J_m}{①} - \underset{J_f}{②} - 3 - \underset{J_f}{④} - 5 \qquad 0 \to {}_{1}{}_{2}{}^{\cdot 4} \to {}_{1}{}_{2}^{2 \cdot 4}{}_{\cdot 4} \to {}_1{}_2^{2 \cdot 4}{}_{\cdot 4} \to \underset{I_1}{{}_1 2^{\cdot 4}} \to 0$

$\mu^+_1 \circ \mu^+_1(DA) = {}_{1}{}_{2}{}^{\cdot 4} \oplus I_2 \oplus I_4$

<u>Sketch</u> Π is the Aus. alg. of $\mathcal{T} := \underline{\operatorname{CM}} R$, where R is a simple singularity R of dim 2

$A = \operatorname{End}_{\mathcal{T}}(\overset{\exists}{} X) \qquad$ Construct $\mu^{+2}_{J_m}(DA)$ by using approximation triangles

Use 1-CY and 2-periodicity $[2] = \mathrm{id}$ of $\mathcal{T}$ $\quad \square$

Def A : CM alg. with dualizing mod. W , $d = \mathrm{fin.dim}\, A$

$\mathrm{CM}A := W^{\perp_{>0}} = \{ X \in \mathrm{mod}A \mid \forall i > 0,\ \mathrm{Ext}^i_A(X, W) = 0 \}$

Fact ⓪ $(\mathrm{CM}A, I^{<\infty}(A))$: cotorsion pair corresp. to W via AR corresp.

① $\mathrm{CM}A \supset \Omega^d(\mathrm{mod}A)$

② $\mathrm{CM}A = \Omega^d(\mathrm{mod}A) \Leftarrow A$ is Iwanaga-Gorenstein (i.e. $W = A$)

Question [AR] Does the converse of ② hold ?

Answer $\exists$ many counterexamples

Thm [CIM] Assume

① $J_f = c(J_f) \subsetneq J$

② $\forall$ conn. comp. C of $\Delta \setminus J_f$, at least one of $C \cap J_m$ and $c(C) \cap J_m$ is ϕ

$\Rightarrow$ $\mathrm{fin.dim}\, A = 2 = \mathrm{dom.dim}\, A$ $\Rightarrow$ $\mathrm{CM}A = \Omega^2(\mathrm{mod}A)$

If moreover $(\Delta, J_f) \neq (A_{2n-1}, \{n\})$, then A is not Iwanaga-Gorenstein