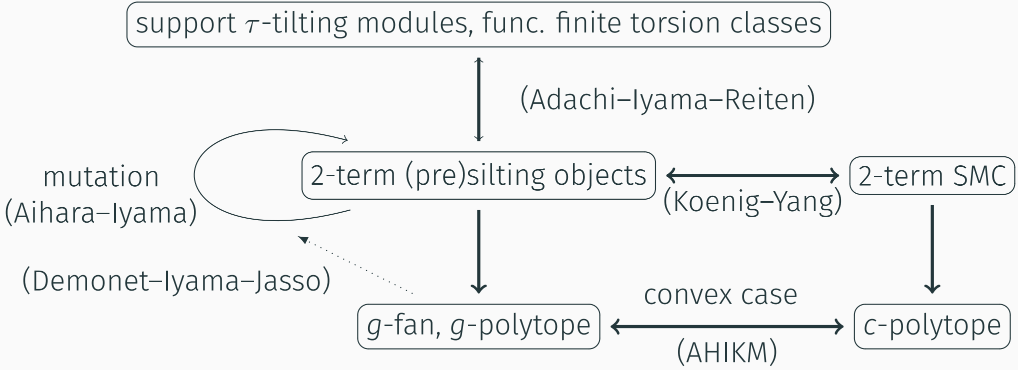
g-fans of rank 2

Toshitaka Aoki, Akihiro Higashitani, Osamu Iyama, Ryoichi Kase, and Yuya Mizuno



Aim

Give a classification of complete g-fans of rank 2.

In this talk, A is a finite dimensional algebra over a field k.

2-term (pre)silting objects

Definition (2-term (pre)silting objects)

T : 2-term object in $\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$, i.e., $T \simeq [\cdots \to 0 \to \overset{-1\text{st}}{P'} \to \overset{0\text{th}}{P} \to 0 \to \cdots]$.

(1) T is 2-term *presilting* if $\mathrm{Hom}_{\mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)}(T, T[1]) = 0$.

(2) T is 2-term *silting* if T is presilting and $|T| = |A|$.

$$\begin{aligned} \text{2-psilt}A &:= \{\text{basic 2-term presilting object of } A\}(/\simeq) \\ \text{2-silt}A &:= \{\text{basic 2-term silting object of } A\}(/\simeq) \end{aligned}$$

Simplicial complex $\Delta(A)$

2-psilt$A$ gives rise to a simplicial complex $\Delta(A)$.

g-vector

Fix a complete set of indecomposable projective modules $P_1, \dots, P_n$.

$$\begin{aligned} K_0(\operatorname{proj} A) &:= \text{Grothendieck group of } \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A) \\ K_0(\operatorname{proj} A)_{\mathbb{R}} &:= K_0(\operatorname{proj} A) \otimes_{\mathbb{Z}} \mathbb{R} : \text{ Real Grothendieck group of } \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A) \end{aligned}$$

$([P_1], \dots, [P_n])$: $\mathbb{Z}$-basis of $K_0(\operatorname{proj} A) \rightsquigarrow K_0(\operatorname{proj} A) = \mathbb{Z}^n$, $K_0(\operatorname{proj} A)_{\mathbb{R}} = \mathbb{R}^n$

Definition (g-vectors)

We say that $[T] \in K_0(\operatorname{proj} A) = \mathbb{Z}^n$ is the *g-vector* of $T \in \mathrm{K}^{\mathrm{b}}(\operatorname{proj} A)$.

If $T = \left[\bigoplus_i P_i^{\oplus b_i} \to \bigoplus_i P_i^{\oplus a_i}\right]$, then we have

$$[T] = [a_1 - b_1 \; a_2 - b_2 \; \cdots \; a_n - b_n]$$

Theorem (Adachi–Iyama–Reiten)

If $[T] = [T']$ holds for 2-term presilting objects T and T', then $T \simeq T'$.

Theorem (Aihara–Iyama)

Let $T = T_1 \oplus \cdots \oplus T_n \in$ 2-silt$A$. Then $\{[T_1], \ldots, [T_n]\}$ forms a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

g-vector

Let $A = k[1 \xrightarrow{a} 2]$ be a path algebra of type A_2. The complete set of indecomposable 2-term presilting objects of A is given by

$$P_1 = e_1A,\ P_2 = e_2A,\ P_a := [P_2 \xrightarrow{a\cdot -} P_1],\ P_1[1],\ P_2[1].$$

The corresponding g-vectors are

$$[1\ 0],\ [0\ 1],\ [1\ -1],\ [-1\ 0],\ [0\ -1]$$

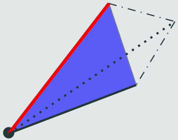
Furthermore, we have

$$\text{2-silt}A = \{P_1 \oplus P_2,\ P_1 \oplus P_a,\ P_a \oplus P_2[1],\ P_2 \oplus P_1[1],\ P_1[1] \oplus P_2[1]\}.$$

Fans in $\mathbb{R}^n$

Definition (Cones in $\mathbb{R}^n$)

For $V \subseteq \mathbb{Q}^n$, we define $\operatorname{cone} V := \sum_{\mathbf{x} \in V} \mathbb{R}_{\geq 0}\mathbf{x} \subseteq \mathbb{R}^n$. We say that $\sigma = \operatorname{cone} V$ is a *strongly convex rational polyhedral cone* (cone for short) in $\mathbb{R}^n$ if $\sigma \cap (-\sigma) = \{\mathbf{0}\}$ holds.



Definition (Fans in $\mathbb{R}^n$)

A fan Σ in $\mathbb{R}^n$ is a collection of cones in $\mathbb{R}^n$ satisfying the following conditions:

- Each face of a cone in Σ is also contained in Σ.
- The intersection of two cones in Σ is a face of each of those two cones.

- Σ is *complete* if $|\Sigma| := \bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$ holds.
- Σ is *nonsingular* if each maximal cone in Σ is generated by a $\mathbb{Z}$-basis of $\mathbb{Z}^n$.

g-fans

Associated with $U = X_1 \oplus \cdots \oplus X_d \in$ 2-psilt$A$, we define a cone $C(U)$ in $\mathbb{R}^n$ as follows:

$$C(U) := \mathrm{cone}\{[X_1], \dots, [X_d]\}$$

Then we set

$$\Sigma(A) := \{C(U) \mid U \in \text{2-psilt}A\}.$$

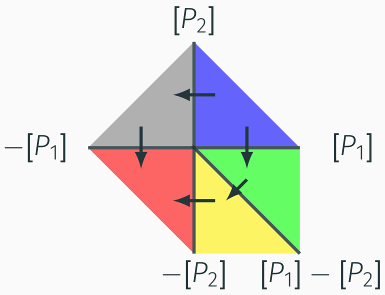
Definition (g-fans of rank n, Demonet–Iyama–Jasso)

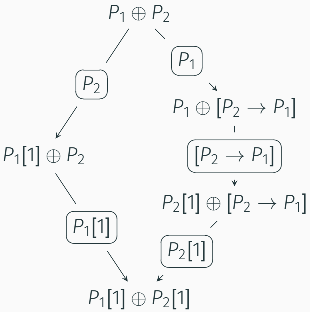
$\Sigma(A)$ is a nonsingular fan in $\mathbb{R}^n$, which we call a *g-fan*.

$\Sigma(A)$	$\Delta(A)$
$\Sigma(A)_d :=$ {Cones of dimension d in $\Sigma(A)$}	$\text{2-psilt}^d A := \{U \in \text{2-psilt}A \mid \lvert U\rvert = d\}$
Faces of $C(U)$	Direct summands of U
The intersection of $C(U)$ and $C(U')$	The maximal common direct summands of U and U'

$\Sigma(A)$	$\Delta(A)$
$\Sigma(A)_d :=$ {Cones of dimension d in $\Sigma(A)$}	$\text{2-psilt}^d A := \{U \in \text{2-psilt}A \mid \lvert U\rvert = d\}$
Faces of $C(U)$	Direct summands of U
The intersection of $C(U)$ and $C(U')$	The maximal common direct summands of U and U'

Let $A = k[1 \xrightarrow{a} 2]$. Then we have the following pictures:





Problem and our result

Problem

Characterize fans of rank n which can be realized as g-fans of some finite dimensional algebras.

Theorem (Aoki–Higashitani–Iyama–K–Mizuno)

Complete g-fans of rank 2 are precisely complete sign-coherent fans of rank 2.

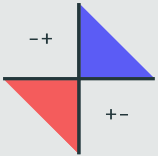
In the rest of this talk, we explain the above theorem.

Sign-coherent fans

Definition (Sign-coherent fans of rank 2)

A nonsingular fan Σ in $\mathbb{R}^2$ is a *sign-coherent* fan of rank 2 if the following conditions holds:

- $\sigma_+ := \mathbb{R}_{\geq 0}[1\ 0] + \mathbb{R}_{\geq 0}[0\ 1] \in \Sigma$, $\sigma_- := -\sigma_+ \in \Sigma$.
- Each cone of dimension 1 is a face of precisely two cones of dimension 2.



Proposition (Adachi–Iyama–Reiten, AHIKM)

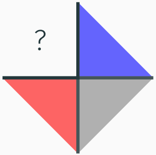
$\Sigma(A)$ is a sign-coherent fan equipped with $\sigma_+ = C(A)$ and $\sigma_- = C(A[1])$

Let $\text{c-Fan}_{\text{sc}}(2)$ the set of complete sign-coherent fans of rank 2. All fans in $\text{c-Fan}_{\text{sc}}(2)$ can be obtained by *subdivision method* and *gluing method*.

To explain these methods, we define $\text{c-Fan}^{+-}_{\text{sc}}(2)$, $\text{c-Fan}^{-+}_{\text{sc}}(2)$ as follows:

$$\begin{aligned} \text{c-Fan}^{+-}_{\text{sc}}(2) &:= \{\Sigma \in \text{c-Fan}_{\text{sc}}(2) \mid \text{cone}\{[-1,0],[0,1]\} \in \Sigma\} \\ \text{c-Fan}^{-+}_{\text{sc}}(2) &:= \{\Sigma \in \text{c-Fan}_{\text{sc}}(2) \mid \text{cone}\{[1,0],[0,-1]\} \in \Sigma\} \end{aligned}$$

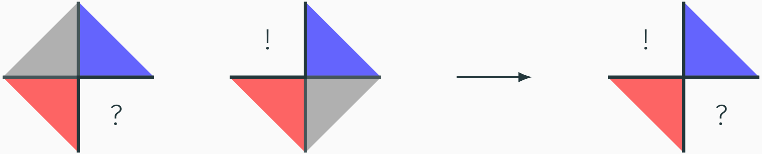
$\text{c-Fan}^{+-}_{\text{sc}}(2)$



$\text{c-Fan}^{-+}_{\text{sc}}(2)$

Gluing method

Let $\Sigma \in \text{c-Fan}_{\mathrm{sc}}^{+-}(2)$ and $\Sigma' \in \text{c-Fan}_{\mathrm{sc}}^{-+}(2)$. We define a new fan $\Sigma * \Sigma'$ as follows:



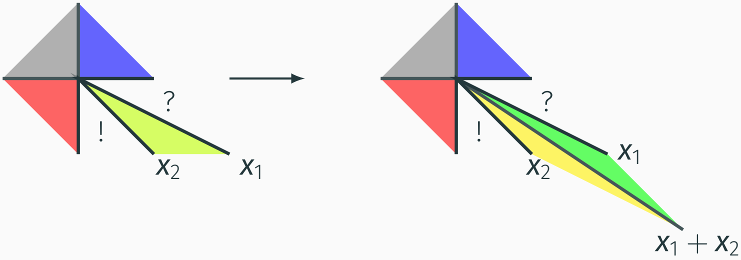
Σ $\quad$ Σ' $\quad$ $\Sigma * \Sigma'$

Construction of $\text{c-Fan}_{\mathrm{sc}}(2)$ via gluing method

$$\text{c-Fan}_{\mathrm{sc}}(2) = \text{c-Fan}_{\mathrm{sc}}^{+-}(2) * \text{c-Fan}_{\mathrm{sc}}^{-+}(2)$$

Subdivision method

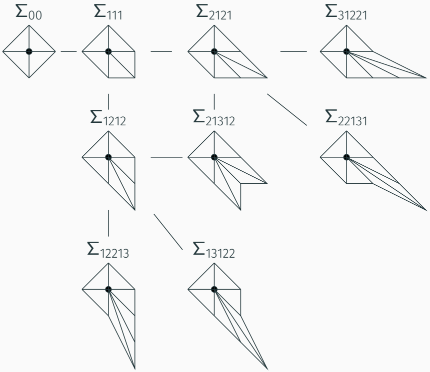
From the $\mathbb{Z}$-basis $\{x_1, x_2\}$ of $\mathbb{Z}^2$, two $\mathbb{Z}$-bases $\{x_1, x_1 + x_2\}$ and $\{x_1 + x_2, x_2\}$ can be formed. This construction allows a cone σ to be divided into two cones. In particular, a new fan $D_\sigma(\Sigma)$ can be constructed from the nonsingular fan Σ and a cone σ in Σ.



Subdivision method

Construction of $\text{c-Fan}^{+-}_{\mathrm{sc}}(2)/\text{c-Fan}^{-+}_{\mathrm{sc}}(2)$ via subdivision method

Any fan in $\text{c-Fan}^{+-}_{\mathrm{sc}}(2)/\text{c-Fan}^{-+}_{\mathrm{sc}}(2)$ can be obtained by repeatedly subdividing the following fan  in the fourth/second quadrant.



Theorem (AHIKM)

Complete g-fans of rank 2 are precisely complete sign-coherent fans of rank 2.

For the proof, we realized gluing and subdivision as algebraic operations. (For simplicity in the following description, we assume k is an algebraically closed field.)

Theorem (Gluing theorem and subdivision theorem, AHIKM)

(1) *Let A, B be k-algebras such that* $\Sigma(A) \in \text{c-Fan}^{+-}_{\mathrm{sc}}(2)$ *and* $\Sigma(B) \in \text{c-Fan}^{-+}_{\mathrm{sc}}(2)$. *Then we can construct an algebra $A * B$ satisfying* $\Sigma(A * B) = \Sigma(A) * \Sigma(B)$.

(2) *Let A be an k-algebra such that* $\Sigma(A) \in \text{c-Fan}^{+-}_{\mathrm{sc}}(2)$ *and σ be a maximal cone in* $\mathbb{R}_{\geq 0}\begin{bmatrix}1 & 0\end{bmatrix} + \mathbb{R}_{\geq 0}\begin{bmatrix}0 & -1\end{bmatrix}$. *Then we can construct an algebra $D_\sigma(A)$ satisfying* $\Sigma(D_\sigma(A)) = D_\sigma(\Sigma(A))$.

Construction of $A * B$

$$A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix}, \; B = \begin{bmatrix} B_1 & 0 \\ Y & B_2 \end{bmatrix}$$

- A_i, B_i: local algebras $\rightsquigarrow A_i/J(A_i) = k$, $B_i/J(B_i) = k$.
- $\overline{(\bullet)} : A_i, B_i \to k$: canonical surjections.
- $C_i := A_i \times_k B_i = \{(a, b) \in A \times B \mid \overline{a} = \overline{b}\}$ (local algebra)
- X: A_1-A_2 bimodule $\rightsquigarrow$ C_1-C_2 bimodule
- Y: B_2-B_1 bimodule $\rightsquigarrow$ C_2-C_1 bimodule

Then we define $A * B$ as follows:

$$A * B = \begin{bmatrix} C_1 & X \\ Y & C_2 \end{bmatrix} \text{ (multiplicatios } X \times Y \to C_1 \text{ and } Y \times X \to C_2 \text{ are 0)}$$

Construction of $A * B$

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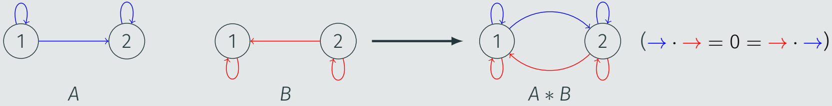
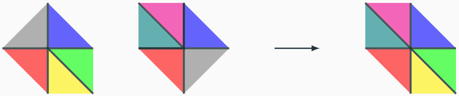


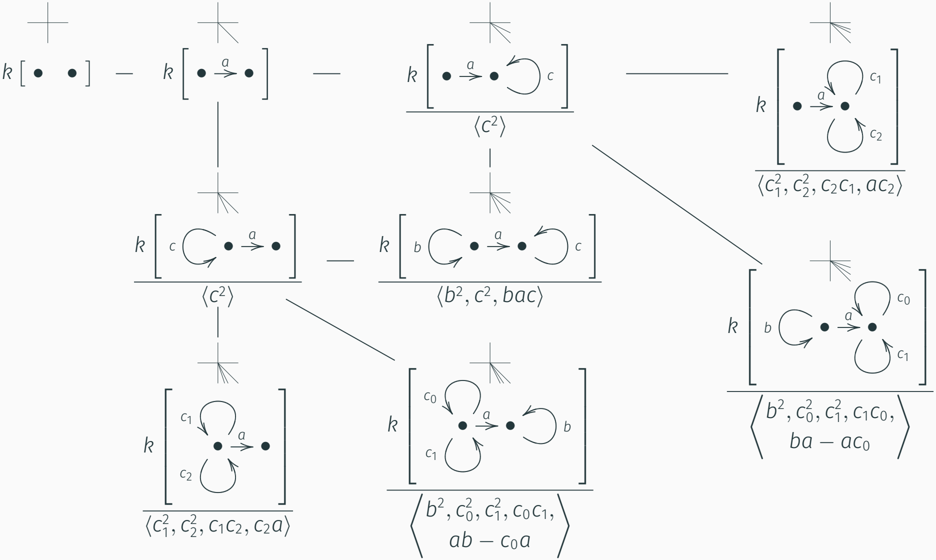
Let $A = k[1 \xrightarrow{a} 2] \cong \begin{bmatrix} k & k \\ 0 & k \end{bmatrix}$ and $B = k[1 \xleftarrow{b} 2] \cong \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$. Then we have

$$A * B = k\left[\, 1 \underset{b}{\overset{a}{\rightleftarrows}} 2 \,\right] / \langle ab, ba \rangle$$



$\Sigma(A)$ $\Sigma(B)$ $\Sigma(A * B) = \Sigma(A) * \Sigma(B)$

Subdivision theorem (examples)



Questions

- Is there a way to directly construct an algebra from a given complete sign-coherent fan of rank 2?
- What happens if the rank is 3 or higher?