Classification of locally free sheaf bimodules of rank 2 over a projective line

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Contents:

$k = \overline{k}$, $\mathrm{char} k = 0$.

X, Y : smooth projective schemes over k.

scheme $\longrightarrow$ ring

sheaf $\longrightarrow$ module

Based on a joint work with Shinnosuke Okawa and Kazushi Ueda (MOU).

1 Motivations

Let A be a noetherian graded algebra.

- $\mathrm{grmod} A$: the category of finitely generated graded right A-modules.
- $\mathrm{GKdim}\, M := \min \left\{ d \in \mathbb{N} \,\middle|\, \lim_{n\to\infty} \frac{\sum_{i=0}^{n} \dim_k M_i}{n^d} \text{ converges} \right\}$.

For $i \in \mathbb{N}$, $(\mathrm{grmod} A)_i := \{M \in \mathrm{grmod} A \mid \mathrm{GKdim}\, M \leq i\}$ is a Serre subcategory of $\mathrm{grmod} A$.

$\mathrm{Proj}_{\mathrm{nc}} A := \mathrm{grmod} A/(\mathrm{grmod} A)_0$ is the noncommutative projective scheme associated to A.

$\mathrm{Proj}_{\mathrm{nc}} A$ is a noncommutative (integral) surface if $\mathrm{GKdim} A = 3$ (and A is a domain).

$\mathrm{Proj}_{\mathrm{nc}} A$ and $\mathrm{Proj}_{\mathrm{nc}} B$ are birationally equivalent $:\Longleftrightarrow$

(1) $\mathrm{GKdim} A = \mathrm{GKdim} B = d$, and

(2) $\mathrm{grmod} A/(\mathrm{grmod} A)_{d-1} \cong \mathrm{grmod} B/(\mathrm{grmod} B)_{d-1}$.

Artin's conjecture (1997)

Every noncommutative integral surface Z is birationally equivalent to either

(1) a noncommutative projective plane (classified),

(2) a noncommutative ruled surface, or

(3) a noncommutative surface finite over its center.

Aim

To classify noncommutative ruled surfaces $\mathbb{P}_X(\mathcal{E})$:

(1) Classify commutative curves X (classified).

(2) Classify locally free sheaf bimodule $\mathcal{E}$ of rank 2 over each commutative curve X.

Focus on the case $X = \mathbb{P}^1$.

2 Commutative $\mathbb{P}^1$-bundles

- $\operatorname{Mod} X$: the category of quasi-coherent sheaves on X.
- $\operatorname{mod} X$: the category of coherent sheaves on X.

If R is a noetherian commutative ring and $X = \operatorname{Spec} R$, then

- $\operatorname{Mod} X \cong \operatorname{Mod} R$: the category of R-modules.
- $\operatorname{mod} X \cong \operatorname{mod} R$: the category of finitely generated R-modules.

Definition

(1) Z is a $\mathbb{P}^1$-bundle over X $:\Longleftrightarrow \exists \mathcal{E} \in \operatorname{mod} X$ locally free of rank 2 s.t. $Z \cong \mathbb{P}_X(\mathcal{E}) := \operatorname{Proj} S_X(\mathcal{E})$ where $S_X(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over X.

(2) Z is a ruled surface $:\Longleftrightarrow Z$ is a $\mathbb{P}^1$-bundle over a curve X.

(3) Z is a Hirzebruch surface $:\Longleftrightarrow Z$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$\mathcal{E} \in \operatorname{mod} X$ is locally free of rank 2 $:\Leftrightarrow \mathcal{E}_p \cong \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p}\ \forall p \in X$

Example

If $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X$, then

$$\mathbb{P}_X(\mathcal{E}) := \mathrm{Proj} S_X(\mathcal{O}_X \oplus \mathcal{O}_X) = \mathrm{Proj} \mathcal{O}_X[x, y] \cong X \times \mathbb{P}^1.$$

In particular, if R is a commutative ring, $X = \operatorname{Spec} R$ and $\mathcal{E} = \mathcal{O}_X \oplus \mathcal{O}_X = R \oplus R$, then

$$\mathbb{P}_X(\mathcal{E}) := \mathrm{Proj} S_X(\mathcal{O}_X \oplus \mathcal{O}_X) = \mathrm{Proj} S_R(R \oplus R) \cong \mathrm{Proj} R[x, y].$$

$\mathcal{L} \in \operatorname{mod} X$ is invertible $:\Longleftrightarrow - \otimes_X \mathcal{L} : \operatorname{Mod} X \to \operatorname{Mod} X$ is an autoequivalence

$\operatorname{Pic} X := \{\mathcal{L} \in \operatorname{mod} X \mid \mathcal{L}$ is invertible$\}$

Theorem

Let $\mathcal{E}, \mathcal{E}' \in \operatorname{mod} X$ be locally free of rank 2.
$\exists \mathcal{L} \in \operatorname{Pic} X$ s.t. $\mathcal{E}' \cong \mathcal{E} \otimes_X \mathcal{L} \Longleftrightarrow \mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$.

Lemma

(1) $\operatorname{Pic} \mathbb{P}^1 = \{\mathcal{O}_{\mathbb{P}^1}(a) \mid a \in \mathbb{Z}\}$.

(2) $\mathcal{E} \in \operatorname{mod} \mathbb{P}^1$ is locally free of rank 2 $\Longleftrightarrow \exists a, b \in \mathbb{Z}$ s.t. $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$.

Corollary

Z is a Hirzebruch surface $\Longleftrightarrow Z \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))\ \exists d \in \mathbb{N}$.

Definition

$\mathbb{F}_d := \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(d))$: Hirzebruch surface of degree d.

$\mathbb{F}_0 = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \cong \mathbb{P}^1 \times \mathbb{P}^1$.

3 Noncommutative $\mathbb{P}^1$-bundles

Let R, S be commutative rings.

- $\operatorname{Mod} R$: the category of R-modules
- $\operatorname{BiMod}(R\text{-}S)$: the category of R-S bimodules

There are two ways to characterize an R-S bimodule:

(a) $\operatorname{BiMod}(R\text{-}S) \cong \operatorname{Mod}(R \otimes S) \cong \operatorname{Mod}(\operatorname{Spec}(R \otimes S)) \cong \operatorname{Mod}(\operatorname{Spec} R \times \operatorname{Spec} S)$

(b) $\operatorname{BiMod}(R\text{-}S) \cong \{- \otimes_R M : \operatorname{Mod} R \rightleftarrows \operatorname{Mod} S : \operatorname{Hom}_S(M, -) \mid$ adjoint pair of functors$\}$

Definition

(a) Let $\mathcal{E} \in \operatorname{mod}(X \times Y)$, and
$W := \operatorname{Supp} \mathcal{E} = \{p \in X \times Y \mid \mathcal{E}_p \neq 0\} \subset X \times Y$.
$\mathcal{E}$ is a sheaf X-Y bimodule if the restrictions of the projections $u := pr_1|_W : W \to X, v := pr_2|_W : W \to Y$ are both finite.

(b) $\mathcal{E}$ is an X-Y bimodule if $\mathcal{E}$ is an adjoint pair of functors

$$- \otimes_X \mathcal{E} : \operatorname{Mod} X \rightleftarrows \operatorname{Mod} Y : \mathcal{H}om_Y(\mathcal{E}, -).$$

- $\operatorname{bimod}(X\text{-}Y)$: the category of sheaf X-Y bimodules
- $\operatorname{BiMod}(X\text{-}Y)$: the category of X-Y bimodules

Theorem [Van den Bergh (2012)]

There exists a fully faithful functor $\operatorname{bimod}(X\text{-}Y) \to \operatorname{BiMod}(X\text{-}Y)$.

$\mathcal{E} \mapsto - \otimes_X \mathcal{E} := pr_{2*}(pr_1^*(-) \otimes_{X \times Y} \mathcal{E})$ (Fourier-Mukai transform)

Definition

(1) Z is a noncommutative $\mathbb{P}^1$-bundle over X $:\Longleftrightarrow$ $\exists\mathcal{E} \in \mathrm{bimod}(X\text{-}X)$ locally free of rank 2 s.t. $Z \cong \mathbb{P}_X(\mathcal{E}) := \mathrm{Proj}_{\mathrm{nc}} S_X(\mathcal{E})$ where $\mathrm{Proj}_{\mathrm{nc}} S_X(\mathcal{E})$ is the noncommutative projective scheme associated to the "noncommutative symmetric algebra" $S_X(\mathcal{E})$ of $\mathcal{E}$ over X.

(2) Z is a noncommutative ruled surface $:\Longleftrightarrow$ Z is a noncommutative $\mathbb{P}^1$-bundle over a curve X.

(3) Z is a noncommutative Hirzebruch surface $:\Longleftrightarrow$ Z is a noncommutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$.

$\mathcal{E} \in \mathrm{bimod}(X\text{-}X)$ is locally free of rank 2 $:\Longleftrightarrow$
$(\mathcal{O}_X \otimes_X \mathcal{E})_p \cong \mathcal{O}_{X,p} \oplus \mathcal{O}_{X,p}$ $\forall p \in X$

Example

The only locally free sheaf of rank 2 on $X = \operatorname{Spec} k$ is k^2, so the $\mathbb{P}^1$-bundle over $X = \operatorname{Spec} k$ is

$$\mathbb{P}_k(k^2) = \mathrm{Proj} S_k(k^2) \cong \mathrm{Proj} k[x,y] \cong \mathbb{P}^1 \cong \mathrm{Proj}_{\mathrm{nc}} \Pi(\bullet \rightrightarrows \bullet)$$

where $\Pi(\bullet \rightrightarrows \bullet)$ is the "preprojective algebra" of the 2-Kronecker quiver.

$S_X(\mathcal{E}) \cong \Pi\left(\mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X\right)$ is the "preprojective algebra" of a quiver $\mathcal{O}_X \xrightarrow{\mathcal{E}} \mathcal{O}_X$

Theorem

Let $\mathcal{E}, \mathcal{E}' \in \mathrm{bimod}(X\text{-}X)$ be locally free of rank 2.
$\exists \mathcal{L}_1, \mathcal{L}_2 \in \mathrm{bimod}(X\text{-}X)$ invertible s.t. $\mathcal{E}' \cong \mathcal{L}_1 \otimes_X \mathcal{E} \otimes_X \mathcal{L}_2 \Longrightarrow \mathbb{P}_X(\mathcal{E}) \cong \mathbb{P}_X(\mathcal{E}')$. (Converse??)

$\mathcal{L} \in \mathrm{bimod}(X\text{-}X)$ is invertible $:\Longleftrightarrow - \otimes_X \mathcal{L} : \mathrm{Mod}\, X \to \mathrm{Mod}\, X$ is an autoequivalence

4 Classification

Setup

$\mathcal{E} \in \mathrm{bimod}(X\text{-}Y)$ locally free of rank 2,

$\iota : W := \mathrm{Supp}\,\mathcal{E} \to X \times Y$ embedding,

$u := pr_1|_W : W \to X, v := pr_2|_W : W \to Y$.

- $\mathrm{CM}(W) := \{\mathcal{U} \in \mathrm{mod}\, W \mid \mathcal{U}$ is maximal Cohen-Macaulay$\}$

Lemma

$\exists! \mathcal{U} \in \mathrm{CM}(W)$ s.t. $\iota_* \mathcal{U} \cong \mathcal{E}$.

Aim

Classify $(W, \mathcal{U} \in \mathrm{CM}(W))$ instead of $\mathcal{E} \in \mathrm{bimod}(X\text{-}Y)$.

Theorem [MOU]

If $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_* \mathcal{U} \cong \mathcal{E}$, then one of the following cases occur:

W	u, v	$\mathcal{U}$
integral	$u, v:$ isom.	$\operatorname{rank} \mathcal{U} = 2$
integral	$\deg u = \deg v = 2$	$\operatorname{rank} \mathcal{U} = 1$
$W = W_1 \cup W_2$ reduced	$u\|_{W_i}, v\|_{W_i}:$ isom.	$\operatorname{rank} (\mathcal{U}\|_{W_i}) = 1$
irreducible, non-reduced	$u\|_{W_{red}}, v\|_{W_{red}}:$ isom.	$\operatorname{rank} (\mathcal{U}\|_{W_{red}}) = 1$

Except for the second case, $X \cong Y$.

From now on, we focus on the case $X = Y = \mathbb{P}^1$.

For $W, W' \subset \mathbb{P}^1 \times \mathbb{P}^1$, we define
$W \sim W' :\Longleftrightarrow \exists \tau_1, \tau_2 \in \mathrm{Aut}\mathbb{P}^1$ s.t. $(\tau_1 \times \tau_2)(W) = W'$.

Lemma

Let $\mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ be locally free of rank 2, $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$.
$W' \sim W \Longrightarrow \exists \mathcal{E}' \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ locally free of rank 2 s.t.

$$\mathrm{Supp}\,\mathcal{E}' = W' \text{ and } \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E}').$$

Aim

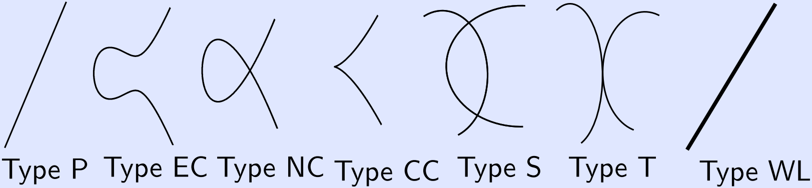
To classify noncommutative Hirzebruch surfaces $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$:

(1) Classify $W \subset \mathbb{P}^1 \times \mathbb{P}^1$ up to $\sim$.

(2) Classify $\mathcal{U} \in \mathrm{CM}(W)$ such that $\iota_*\mathcal{U} \in \mathrm{bimod}(\mathbb{P}^1 \times \mathbb{P}^1)$ is locally free of rank 2 for each W.

Theorem [Patrick (1997), MOU]

$\forall \mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ locally free of rank 2, $W = \mathrm{Supp}\,\mathcal{E} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is a Cartier divisor of bidegree $(1,1)$ or $(2,2)$. In fact, it is equivalent to one of the following types:



If W is not of Type P, then
$\mathcal{U} \in \mathrm{Pic}\,W \subset \mathrm{CM}(W) \Rightarrow \iota_*\mathcal{U} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ is locally free of rank 2.

$Type$	EC	NC	CC	S	T	WL
$\mathrm{Pic}\,W$	$W \times \mathbb{Z}$	$k^\times \times \mathbb{Z}$	$k \times \mathbb{Z}$	$k^\times \times \mathbb{Z}$	$k \times \mathbb{Z}$	$k \times \mathbb{Z}$

Type P

Since $W \sim \Delta_{\mathbb{P}^1} := \{(p,p) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid p \in \mathbb{P}^1\}$, $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \cong \mathbb{F}_d \; \exists d \in \mathbb{N}$ (commutative Hirzebruch surface)

For the rest of the types, we define the non-invertible locus of $\mathcal{U} \in \mathrm{CM}(W)$ by

$$\mathrm{Ninv}(\mathcal{U}) = \{p \in W \mid \mathcal{U}_p \not\cong \mathcal{O}_{W,p}\} \subset \mathrm{Sing}(W).$$

We classify $\mathcal{U} \in \mathrm{CM}(W)$ by analyzing $\mathrm{IndCM}(\mathcal{O}_{W,p})$ (or $\mathrm{IndCM}(\widehat{\mathcal{O}_{W,p}})$) for $p \in \mathrm{Ninv}(\mathcal{U})$.

Type EC (smooth)

Since $\mathrm{Sing}(W) = \emptyset$, $\mathcal{U} \in \mathrm{CM}(W) = \mathrm{Pic}\, W \cong W \times \mathbb{Z}$.

Type NC, CC, S, T (singular, reduced)

For $p \in \mathrm{Sing}(W)$, $\widehat{\mathcal{O}_{W,p}} \cong k[[x,y]]/(y^2 - x^{n+1})$ for $n = 1, 2, 3$. Using the classifications of $\mathrm{IndCM}(\widehat{\mathcal{O}_{W,p}})$, we can show that $\mathcal{U}_p \cong \mathrm{End}_{\mathcal{O}_{W,p}}(\mathcal{U}_p)$ viewed as an $\mathrm{End}_{\mathcal{O}_{W,p}}(\mathcal{U}_p)$-module.

Theorem [MOU]

If $\mathcal{U} \notin \mathrm{Pic}\, W$, then $\exists \widetilde{\mathcal{U}} \in \mathrm{Pic}\, \widetilde{W}$ s.t. $\nu_* \widetilde{\mathcal{U}} \cong \mathcal{U}$ where $\nu : \widetilde{W} := \mathcal{S}pec\, \mathcal{E}nd_W(\mathcal{U}) \to W$.

Type	$\widetilde{W}$	$\mathrm{Pic}\, \widetilde{W}$
NC, CC	$\mathbb{P}^1$	$\mathbb{Z}$
S, T	$\mathbb{P}^1 \cup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$
S, T	$\mathbb{P}^1 \sqcup \mathbb{P}^1$	$\mathbb{Z} \times \mathbb{Z}$

Type WL (non-reduced)

For $p \in \mathrm{Sing}(W) = W$, $\mathcal{O}_{W,p} \cong k[x,y]_{(x)}/(y^2)$. Using the classification of $\mathrm{IndCM}(\mathcal{O}_{W,p})$, we can show that $\mathcal{U}_p \cong (x^n, y) \lhd k[x,y]_{(x)}/(y^2)$ for some $n \in \mathbb{N}$.

Theorem [MOU]

$\sharp(\mathrm{Ninv}(\mathcal{U})) < \infty$ and $\exists \mathcal{L} \in \mathrm{Pic}\, W \cong k \times \mathbb{Z}$ s.t.

$$0 \to \mathcal{U} \to \mathcal{L} \to \mathcal{O}_{\mathrm{Ninv}(\mathcal{U})} \to 0$$

is exact.

5 Conjecture

Definition

Z is a q-$\mathbb{F}_d$ (noncommutative Hirzebruch surface of degree d) $:\Longleftrightarrow$
$Z \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$ $\exists \mathcal{E} \in \mathrm{bimod}(\mathbb{P}^1\text{-}\mathbb{P}^1)$ s.t.
$\mathcal{O}_{\mathbb{P}^1}(k) \otimes_{\mathbb{P}^1} \mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(k+d)$ $\forall k \in \mathbb{Z}$.

Conjecture (next year?)

Z is q-$\mathbb{F}_0$ $\Longleftrightarrow$ $Z \cong \mathrm{Proj}_{\mathrm{nc}} A$ where A is an AS-regular algebra of dimension 3 and of Gorenstein parameter 2 over the path algebra $A_0 = k(\bullet \rightrightarrows \bullet)$ of the 2-Kronecker quiver??

Example

If A is an AS-regular algebra of dimension 2 and of Gorenstein parameter 1 over $A_0 = k(\bullet \rightrightarrows \bullet)$, then $A \cong \Pi(\bullet \rightrightarrows \bullet)$, so
$\mathrm{Proj}_{\mathrm{nc}} A \cong \mathrm{Proj}_{\mathrm{nc}} \Pi(\bullet \rightrightarrows \bullet) \cong \mathbb{P}^1$.

Theorem [MOU]

There exists a semi-orthogonal decomposition

$$\mathcal{D}^b(\operatorname{mod}\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) = \langle \mathcal{D}^b(\operatorname{mod}\mathbb{P}^1), \mathcal{D}^b(\operatorname{mod}\mathbb{P}^1)\rangle$$

with the "dual gluing functor" $-\otimes^{\mathbf{L}}_{\mathbb{P}^1}\mathcal{E} : \mathcal{D}^b(\operatorname{mod}\mathbb{P}^1) \to \mathcal{D}^b(\operatorname{mod}\mathbb{P}^1)$.

Conjecture (in 2 years?)

$$\mathcal{D}^b(\operatorname{mod}\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})) \cong \mathcal{D}^b\left(\operatorname{mod}\begin{pmatrix}\mathcal{O}_{\mathbb{P}^1} & \mathcal{E}\\ 0 & \mathcal{O}_{\mathbb{P}^1}\end{pmatrix}\right)??$$

Example (Beilinson)

$$\mathcal{D}^b(\operatorname{mod}\mathbb{P}^1) \cong \langle \mathcal{D}^b(\operatorname{mod}k), \mathcal{D}^b(\operatorname{mod}k)\rangle$$

$$\cong \mathcal{D}^b(\operatorname{mod}k(\bullet \rightrightarrows \bullet)) \cong \mathcal{D}^b\left(\operatorname{mod}\begin{pmatrix}k & k^2\\ 0 & k\end{pmatrix}\right).$$