Clifford's theorem in wide subcategories

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1. Introduction

What's the Clifford's theorem?

- representation theory of finite groups
- normal subgroups, restriction functor
- semisimple modules

What's a wide subcategory?

- representation theory of rings
- extension-closed exact abelian subcategories
- semibricks, ring epimorphisms, torsion classes

Clifford's theorem in wide subcategories

Why do we consider it?

$\rightsquigarrow$ This arises from the following perspectives

- brick version of Clifford's theorem [Clifford]
- brick label, τ-tilting theory [Koshio-Kozakai]

2. Wide subcategories

$\mathcal{A}$: abelian cat. , subcat. = full and closed under isom.

Λ : f.d. k-alg. , $\mathrm{mod}\,\Lambda$: the cat. of fin. gen. right Λ-modules.

Def. $\mathcal{W} \subseteq \mathcal{A}$: wide subcategory

:⟺ $\mathcal{W}$ is closed under
- extensions
- kernels
- cokernels

$$\begin{array}{ccccccccc} & & \mathcal{W} & & \mathcal{W} & & \mathcal{W} & & \\ & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccccc} 0 & \to & \mathrm{Ker} f & \to & X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{Cok} f & \longrightarrow & 0 \\ & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \rotatebox{-90}{$\in$} & & \\ & & \mathcal{W} & & \mathcal{W} & & \mathcal{W} & & \mathcal{W} & & \end{array}$$

Rmk. $\begin{cases} \mathcal{W} : \text{abelian category} \\ \mathrm{Ext}^1_{\mathcal{W}}(X, Y) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathcal{A}}(X, Y) \quad {}^\forall X, Y \in \mathcal{W} \end{cases}$

e. g. Serre subcategories are wide subcategories

e. g. $\Lambda \longrightarrow \Gamma$: ring epimorph. satisfying $\mathrm{Tor}_i^{\Lambda}(\Gamma, \Gamma) = 0$

(for example $R \longrightarrow S^{-1}R$: localization)

Then $\mathrm{Mod}\,\Gamma \hookrightarrow \mathrm{Mod}\,\Lambda$: wide subcategory

Prop. Λ : f.d. k-alg. , $\mathcal{W} \subseteq \mathrm{mod}\,\Lambda$: wide subcat.

Then

$\mathcal{W} \subseteq \mathrm{mod}\,\Lambda$: functorially finite

$\Longleftrightarrow$ $^{\exists}\Lambda \longrightarrow \Gamma$: ring epi. s.t. $\begin{cases} \Gamma : \text{f.d. k-alg.},\ \mathrm{Tor}_i^{\Lambda}(\Gamma, \Gamma) = 0 \\ \mathrm{mod}\,\Gamma \xrightarrow{\sim} \mathcal{W} \subseteq \mathrm{mod}\,\Lambda \end{cases}$

Def. $\mathcal{T} \subseteq \mathcal{A}$: torsion class

$:\Leftrightarrow$ $\mathcal{T}$ is closed under

- extensions
- quotients

$$\begin{tikzcd} & \mathcal{T} \arrow[d] & \mathcal{T} \arrow[d] & \mathcal{T} \arrow[d] & \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} & & \mathcal{T} \arrow[d] & \mathcal{T} \arrow[d] & \\ 0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \end{tikzcd}$$

Prop. (Ingalls-Thomas, Marks-Šťovíček)

There are operations $\mathbb{T}$ and α

$$\{\text{wide subcat. of } \mathcal{A}\} \underset{\alpha}{\overset{\mathbb{T}}{\rightleftarrows}} \{\text{torsion classes in } \mathcal{A}\}$$

satisfying $\alpha \circ \mathbb{T} = id$

Serre subcategory
simple module
semisimple module

$\xrightarrow{\text{Generalization}}$

wide subcategory
brick
semibrick

<u>Def.</u> $S \in \mathcal{A}$: brick $:\Longleftrightarrow$ $\mathrm{End}(S)$: division ring

A set of isoclasses of bricks $\mathcal{X} \subseteq \mathcal{A}$ is semibrick
$:\Longleftrightarrow$ $X, Y \in \mathcal{X}$, $X \neq Y \Rightarrow \mathrm{Hom}(X, Y) = 0$

$M \in \mathcal{A}$: semibrick $:\Longleftrightarrow$ ${}^{\exists}\mathcal{X} \subseteq \mathcal{A}$: semibrick s.t. $M \cong \bigoplus_{X \in \mathcal{X}} X^{\oplus n_X}$

<u>e. g.</u>

- simple $\Rightarrow$ brick $\Rightarrow$ indecomposable
- sets of simple modules are semibricks
- semisimple modules are semibricks

Thm. (Ringel)

Suppose $\mathcal{A}$: length abelian category.

$$\{\text{wide subcat. of } \mathcal{A}\} \xleftrightarrow{1:1} \{\text{semibricks in } \mathcal{A}\}/\cong$$

$$\cup\!\!\!\!\mid \qquad\qquad\qquad\qquad \cup\!\!\!\!\mid$$

$$\{\text{Serre subcat. of } \mathcal{A}\} \xleftrightarrow{1:1} \{\text{sets of simple modules}\}/\cong$$

$$\mathcal{W} \longmapsto \operatorname{sim}\mathcal{W}$$

$$\operatorname{Filt}\mathcal{X} \longleftarrow\!\shortmid \mathcal{X}$$

$\operatorname{sim}\mathcal{W}$: the set of isoclasses of simple objects in $\mathcal{W}$

$\operatorname{Filt}\mathcal{X} := \{ M \in \mathcal{A} \mid \exists\, 0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M \text{ s.t. } M_i/M_{i-1} \in \mathcal{X} \ (1 \leq i \leq n) \}$

3. Clifford's theorem

G : finite group , kG : group algebra over k

$N \trianglelefteq G$: normal subgroup

Compare $\mathrm{mod}\,kG$ and $\mathrm{mod}\,kN$

$- \otimes_{kN} kG =: \mathrm{Ind}$: induction functor

$$\mathrm{mod}\,kN \rightleftarrows \mathrm{mod}\,kG$$

Res : restriction functor

Thm. (Clifford)

S : simple kG-module

Then $\mathrm{Res}\, S$: semisimple kN-module

Question. Is it possible to generalize the above to bricks, that is,

$$S \in \mathrm{mod}\, kG : \text{brick} \Rightarrow \mathrm{Res}\, S : \text{semibrick} ?$$

This does not hold in general.
However, it is true under some assumptions.
We formalize Clifford's theorem in wide subcatetgories for that.

4. Main results

$N \trianglelefteq G$: normal

Def. $\mathcal{W} \leq \mathrm{mod}\,kG$: wide subcat.

$(\star)$: $\iff$ $\forall W \in \mathcal{W}$ $\quad k[G/N] \otimes_k W \in \mathcal{W}$

Prop. (1) $\mathrm{Ind}^{-1}(\mathcal{W}) := \{ M \in \mathrm{mod}\,kN \mid \mathrm{Ind}\,M \in \mathcal{W} \}$

: wide subcat. of $\mathrm{mod}\,kN$

(2) If $\mathcal{W}$ satisfies $(\star)$,

then Res induces

$$\begin{array}{ccc} \mathrm{mod}\,kG & \xrightarrow{\mathrm{Res}} & \mathrm{mod}\,kG \\ \cup\!| & & \cup\!| \\ \mathcal{W} & \longrightarrow & \mathrm{Ind}^{-1}(\mathcal{W}) \end{array}$$

Clifford's theorem in wide subcategories

$\mathcal{W} \leq \operatorname{mod} kG$: wide subcat. satisfying $(\star)$

Then Clifford's theorem holds in $\mathcal{W}$:

Thm. (Kozakai-S)

S : simple object in abelian cat. $\mathcal{W}$

Then $\operatorname{Res} S$: semisimple in abelian cat. $\operatorname{Ind}^{-1}(\mathcal{W})$

In particular, $\operatorname{Res} S$: semibrick

Rmk. Applying the above to $\mathcal{W} = \operatorname{mod} kG$,

we obtain the ordinary Clifford's theorem.

Cor. (brick version of Clifford's theorem)

Suppose $\operatorname{char} k = p > 0$ and G/N is a p-group

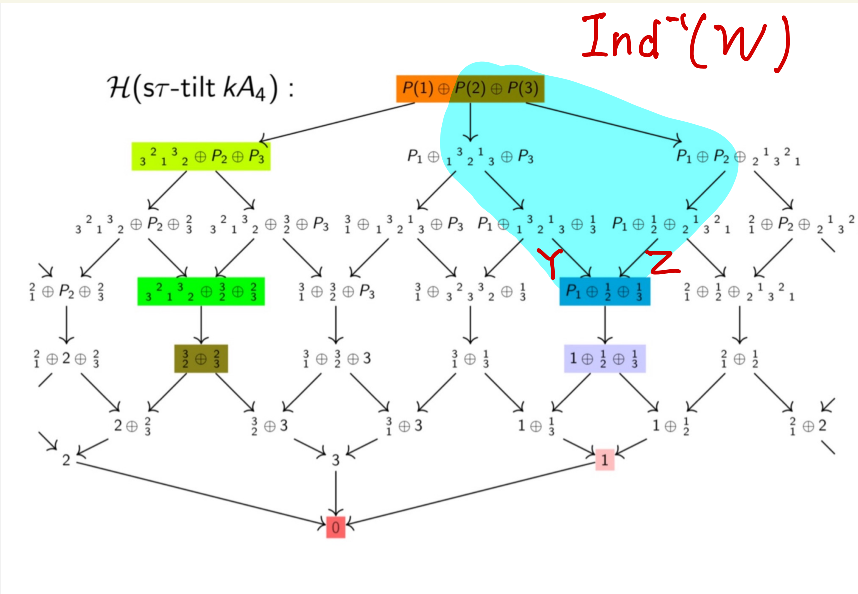
Then for any brick $S \in \operatorname{mod} kG$,

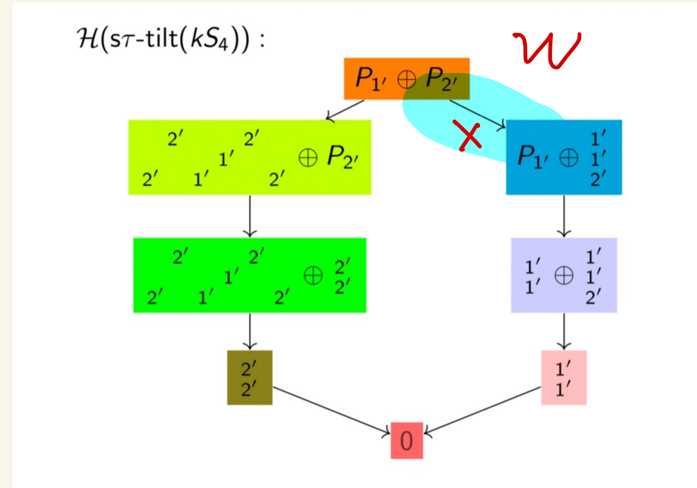
$\operatorname{Res} S$: semibrick

Sketch of a proof.

- S is a simple object in a wide subcat. $\operatorname{Filt} S$ (Ringel)
- $\operatorname{Filt} S$ satisfies (☆) by the assumption.
- Apply the previous theorem.

A perspective from the result by [Koshio-Kozakai]





: wide interval , X, Y, Z : brick label

$\mathrm{Res}\, X \cong Y \oplus Z$ by the main theorem

Suppose $\operatorname{char} k = p > 0$ and G/N is a p-group

$\{X_1, \cdots, X_n\}$: 2-term simple-minded collection in $D^b(\operatorname{mod} kG)$

$\rightsquigarrow$ $X_i \in \operatorname{mod} kG$ or $X_i[-1] \in \operatorname{mod} kG$ (Brüstle-Yang)

$\rightsquigarrow$ $\operatorname{Res} X_i = X_{i1}^{\oplus m} \oplus \cdots \oplus X_{in_i}^{\oplus m}$: semibrick (Cor.)

$\boxed{\begin{array}{l}\underline{\text{Thm.}}\ (\text{Kozakai-S})\\ \{X_{i\alpha}\}_{i,\alpha} : \text{2-term simple-minded collection in } D^b(\operatorname{mod} kN)\end{array}}$