

The stable category of Gorenstein-projective modules over a monomial algebra

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Table of contents

- 1 Motivation
- 2 Stable categories of graded Gorenstien-projective modules
- 3 Stable categories of Gorenstien-projective modules

Notation

- Λ : a finite dimensional algebra over a field K
- $\text{mod } \Lambda$: the category of (finitely generated right) Λ -modules
- $\text{proj } \Lambda$: the category of projective Λ -modules

1. Motivation

Definition (Enochs-Jenda 1995)

M_Λ : **Gorenstein-projective (GP)** $\stackrel{\text{def}}{\iff} \exists$ an acyclic complex of projective Λ -modules

$$P^\bullet : \dots \rightarrow P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \rightarrow \dots$$

s.t. (i) $\text{Hom}_\Lambda(P^\bullet, \Lambda)$ is exact; and (ii) $M \cong \text{Ker } d^0$ as Λ -modules

- P^\bullet : a **complete resolution** of M
- GP modules are called modules of Gorenstein dimension zero (Auslander-Bridger 1969), totally reflexive modules (Avramov-Martsinkovsky 2002), and maximal Cohen-Macaulay modules (Buchweitz 1989)
- $\text{Gproj } \Lambda$: the category of GP Λ -modules $\implies \text{proj } \Lambda \subseteq \text{Gproj } \Lambda \subseteq \text{mod } \Lambda$
- $\text{Gproj } \Lambda = \text{mod } \Lambda \iff \Lambda$: self-injective

Definition

Λ : **CM-free** $\stackrel{\text{def}}{\iff} \text{proj } \Lambda = \text{Gproj } \Lambda$

- $\text{gl.dim } \Lambda < \infty \implies \Lambda$: CM-free

- From now on, assume Λ is monomial (i.e. $\Lambda = KQ/I$, where I is generated by paths) $\implies \Lambda$ is CM-finite (i.e. $\#\text{ind Gproj } \Lambda < \infty$) since any indecomposable non-projective GP Λ -module is of the form $p\Lambda$ for some non-zero non-trivial path p

Definition (Chen-Shen-Zhou 2018)

- A pair (p, q) of non-zero paths in Λ is **perfect** if the following are satisfied:
 - ① p and q are both non-trivial with $t(p) = s(q)$ and satisfy $pq = 0$ in Λ
 - ② If $pq' = 0$ for a non-zero path q' with $t(p) = s(q')$, then $q' = qq''$ for some path q''
 - ③ If $p'q = 0$ for a non-zero path p' with $t(p') = s(q)$, then $p' = p''p$ for some path p''
- $(p_1, \dots, p_n, p_{n+1} = p_1)$: a **perfect path sequence** if (p_i, p_{i+1}) is perfect for $1 \leq i \leq n$
- A path in a perfect path sequence is called a **perfect path**

- \mathbb{P}_Λ : the set of perfect paths

Theorem (CSZ 2018)

$$\begin{array}{ccc} \mathbb{P}_\Lambda & \xleftarrow{1:1} & \{ \text{indecomposable non-projective GP } \Lambda\text{-modules} \} / \cong \\ p & \longmapsto & p\Lambda \end{array}$$

- \mathbb{P}_Λ is empty $\iff \Lambda$ is CM-free

Remark Perfect path sequences give rise to complete resolutions

Definition (Chen-Shen-Zhou 2018)

- A pair (p, q) of non-zero paths in Λ is **perfect** if the following are satisfied:
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- $(p_1, \dots, p_n, p_{n+1} = p_1)$: a **perfect path sequence** if (p_i, p_{i+1}) is perfect for $1 \leq i \leq n$
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Example

Consider the monomial algebra $\Lambda = K \left(1 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} x \right) / (x^5)$

- $(x, x^4, x), (x^2, x^3, x^2)$: the minimal perfect path sequences
- $\mathbb{P}_\Lambda = \{x, x^2, x^3, x^4\}$, hence $\text{ind Gproj } \Lambda = \{x\Lambda, x^2\Lambda, x^3\Lambda, x^4\Lambda\} \cup \text{ind proj } \Lambda$
- For example, (x, x^4, x) induces the exact sequence

$$\begin{array}{ccccccc}
 0 & \longrightarrow & x\Lambda & \longrightarrow & \Lambda & \longrightarrow & \Lambda & \longrightarrow & x\Lambda & \longrightarrow & 0 \\
 & & & & & & \searrow & & \nearrow & & \\
 & & & & & & & x^4\Lambda & & &
 \end{array}$$

- The stable category $\underline{\text{Gproj}} \Lambda = \text{Gproj} \Lambda / \text{proj} \Lambda$ of $\text{Gproj} \Lambda$ carries a structure of a triangulated category

Theorem (CSZ 2018)

TFAE

- 1 $\underline{\text{Gproj}} \Lambda$ is a semisimple triangulated category
- 2 \exists no overlap in Λ
- 3 $\underline{\text{Gproj}} \Lambda \cong \prod_{i=1}^n (\text{mod } k^{n_i}, \sigma^*)$, where the automorphism $\sigma^* : \text{mod } k^{n_i} \rightarrow \text{mod } k^{n_i}$ is induced by $\sigma : k^{n_i} \rightarrow k^{n_i} \in \text{Aut } \Lambda$ given by $\sigma(\lambda_1, \lambda_2, \dots, \lambda_{n_i}) = (\lambda_2, \dots, \lambda_{n_i}, \lambda_1)$

Remark \exists no overlap in $\Lambda \iff$ there exists no non-trivial morphism in $\underline{\text{Gproj}} \Lambda$

- Ringel (2013) and Lu-Zhu (2021) determined $\underline{\text{Gproj}} \Lambda$ for Nakayama algebras and 1-Iwanaga-Gorenstein monomial algebras, respectively

Recall $\Lambda : (d\text{-})\text{Iwanaga-Gorenstein} \xleftrightarrow{\text{def}} \text{id}_\Lambda \Lambda, \text{id } \Lambda_\Lambda \leq d < \infty$

- In any cases, $\underline{\text{Gproj}} \Lambda \cong \underline{\text{mod}} \Gamma$ for some self-injective Nakayama algebra Γ

Our aim is

to describe $\underline{\text{Gproj}} \Lambda$ for more general monomial algebras Λ

Remark Many authors such as Chen-Geng-Lu (2015), Lu (2016, 2019), Enomoto (2018) and Minamoto-Yamaura (2020) describe $\underline{\text{Gproj}} \Lambda$ for specific classes of non-monomial (Iwanaga-Gorenstein) algebras Λ

Theorem (Buchweitz 1986)

If Λ is an Iwanaga-Gorenstein algebra, then $\underline{\text{Gproj}} \Lambda$ is triangle equivalent to the singularity category $\mathcal{D}_{\text{sg}}(\text{mod } \Lambda) := \mathcal{D}^b(\text{mod } \Lambda) / \mathcal{K}^b(\text{proj } \Lambda)$

2. Stable categories of graded Gorenstien-projective modules

Definition

The **underlying cycle** c_p associated with $p \in \mathbb{P}_\Lambda$ is the shortest cycle c s.t. $p_1 \cdots p_n = c^l$ for some $l > 0$, where $(p = p_1, \dots, p_n, p_{n+1} = p_1)$ is a perfect path sequence

- $\mathcal{C}(\Lambda)$: the set of equivalence classes (w.r.t. cyclic permutation) of underlying cycles

Definition

For p and $q \in \mathbb{P}_\Lambda$, we write $p \preceq q$ if $q = pr$ for some path r

- $(\mathbb{P}_\Lambda, \preceq)$ is a poset
- The Hasse quiver $H(\mathbb{P}_\Lambda, \preceq)$ is a disjoint union of linear quivers

Definition

$p \in \mathbb{P}_\Lambda$: **co-elementary** $\stackrel{\text{def}}{\iff} p$ is a sink in $H(\mathbb{P}_\Lambda, \preceq)$

- $\mathbb{E}_\Lambda^{\text{co}}$: the set of co-elementary paths

Example

Let $\Lambda = KQ/I$ be the monomial algebra given by

$$\begin{array}{ccccc} 1 & \xrightarrow{a_1} & 2 & \xrightarrow{b_2} & 4 \\ & \swarrow a_3 & \downarrow a_2 & \curvearrowright a_4 & \\ & & 3 & & \end{array} \quad a_{1231} = a_{23123} = a_4^4 = 0$$

- The following are the minimal perfect path sequences

$$(a_1, a_{231}, a_{23}, a_{123}, a_1), \quad (a_4, a_4^3, a_4), \quad (a_4^2, a_4^2)$$

- $\mathbb{P}_\Lambda = \{a_1, a_{231}, a_{23}, a_{123}, a_4, a_4^2, a_4^3\}$

- $\mathcal{C}(\Lambda) = \{a_{123}, a_4\}$, where $a_{123} = a_{231}$

- $H(\mathbb{P}_\Lambda, \preceq) : \quad a_{123} \longrightarrow a_1 \quad a_{231} \longrightarrow a_{23} \quad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$

- $\mathbb{E}_\Lambda^{\text{co}} = \{a_1, a_{23}, a_4\}$

Proposition-Definition

For $c \in \mathcal{C}(\Lambda)$, $\exists!$ $r_1, \dots, r_n \in \mathbb{E}_\Lambda^{\text{co}}$ s.t. $c = r_1 \cdots r_n$. We denote $|c| := n$.

- Consider $\Lambda = KQ/I$ as a positively graded algebra by defining $\deg a = 1$ for $a \in Q_1$
- $\text{mod}^{\mathbb{Z}}\Lambda$: the category of graded Λ -modules

Recall For $M, N \in \text{mod}^{\mathbb{Z}}\Lambda$, $\text{Hom}_{\Lambda}^{\mathbb{Z}}(M, N) := \{f \in \text{Hom}_{\Lambda}(M, N) \mid f(M_i) \subseteq N_i \text{ for } i\}$

- $\text{proj}^{\mathbb{Z}}\Lambda$: the category of graded projective Λ -modules
- $\text{Gproj}^{\mathbb{Z}}\Lambda$: the category of graded GP Λ -modules
- $\underline{\text{Gproj}}^{\mathbb{Z}}\Lambda = \text{Gproj}^{\mathbb{Z}}\Lambda / \text{proj}^{\mathbb{Z}}\Lambda$: the stable category of $\text{Gproj}^{\mathbb{Z}}\Lambda$
- Lu-Zhu (2021) observed that $\text{ind } \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda = \{p\Lambda(i) \mid p \in \mathbb{P}_{\Lambda}, i \in \mathbb{Z}\}$

Theorem (LZ 2021)

If Λ is Iwanaga-Gorenstein, then $\underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \cong \mathcal{D}^b(\text{mod } H)$ for some hereditary algebra H of finite representation type

Remark They use the triangle equivalence

$$\underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \cong \mathcal{D}_{\text{sg}}(\text{mod}^{\mathbb{Z}}\Lambda) := \mathcal{D}^b(\text{mod}^{\mathbb{Z}}\Lambda) / \mathcal{K}^b(\text{proj}^{\mathbb{Z}}\Lambda)$$

- For $c = r_1 \cdots r_n \in \mathcal{C}(\Lambda)$ with $r_i \in \mathbb{E}_\Lambda^{\text{co}}$, we define

$$\mathbb{P}_\Lambda(c) := \{p \in \mathbb{P}_\Lambda \mid r_1 \preceq p\} \quad \text{and} \quad T_c := \bigoplus_{p \in \mathbb{P}_\Lambda(c)} p\Lambda$$

- Define $T := \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \leq i < l(c)} T_c(i) \in \underline{\text{Gproj}}^{\mathbb{Z}} \Lambda$

Example

Let $\Lambda = KQ/I$ be defined as earlier:

$$\begin{array}{c}
 1 \xrightarrow{a_1} 2 \xrightarrow{b_2} 4 \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} a_4 \\
 \swarrow a_3 \quad \downarrow a_2 \\
 \quad \quad 3
 \end{array}
 \quad a_{1231} = a_{23123} = a_4^4 = 0$$

- $H(\mathbb{P}_\Lambda, \preceq) : \quad a_{123} \longrightarrow a_1 \quad a_{231} \longrightarrow a_{23} \quad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$

- $\mathcal{C}(\Lambda) = \{a_{123}, a_4\}$

$$\implies T = a_1\Lambda \oplus a_{123}\Lambda \oplus a_1\Lambda(1) \oplus a_{123}\Lambda(1) \oplus a_1\Lambda(2) \oplus a_{123}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$$

- $\mathcal{C}(\Lambda) = \{a_{231}, a_4\}$

$$\implies T = a_{23}\Lambda \oplus a_{231}\Lambda \oplus a_{23}\Lambda(1) \oplus a_{231}\Lambda(1) \oplus a_{23}\Lambda(2) \oplus a_{231}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$$

Example (continued)

- The Auslander-Reiten quiver of $\underline{\text{Gproj}}^{\mathbb{Z}} \Lambda$ is given as follows:

$$\begin{array}{ccccccc}
 \cdots & a_1\Lambda(-3) & \cdots & a_{23}\Lambda(-1) & \cdots & a_1\Lambda & \cdots & a_{23}\Lambda(2) \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow \\
 a_{123}\Lambda(-3) & \cdots & a_{231}\Lambda(-1) & \cdots & a_{123}\Lambda & \cdots & a_{231}\Lambda(2) & \cdots
 \end{array}$$

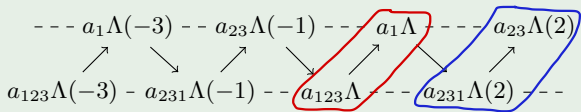
$$\begin{array}{ccccccc}
 \cdots & a_1\Lambda(-2) & \cdots & a_{23}\Lambda & \cdots & a_1\Lambda(1) & \cdots & a_{23}\Lambda(3) \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow \\
 a_{123}\Lambda(-2) & \cdots & a_{231}\Lambda & \cdots & a_{123}\Lambda(1) & \cdots & a_{231}\Lambda(3) & \cdots
 \end{array}$$

$$\begin{array}{ccccccc}
 \cdots & a_1\Lambda(-1) & \cdots & a_{23}\Lambda(1) & \cdots & a_1\Lambda(2) & \cdots & a_{23}\Lambda(4) \\
 & \nearrow & & \searrow & \nearrow & & \searrow & \nearrow \\
 a_{123}\Lambda(-1) & \cdots & a_{231}\Lambda(1) & \cdots & a_{123}\Lambda(2) & \cdots & a_{231}\Lambda(4) & \cdots
 \end{array}$$

$$\begin{array}{ccccccc}
 a_4\Lambda(-2) & \cdots & a_4\Lambda(-1) & \cdots & a_4\Lambda & \cdots & a_4\Lambda(1) \\
 & \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow \\
 \cdots & a_4^2\Lambda(-1) & \cdots & a_4^2\Lambda & \cdots & a_4^2\Lambda(1) & \cdots \\
 & \nearrow & & \searrow & \nearrow & & \searrow \\
 a_4^3\Lambda(-1) & \cdots & a_4^3\Lambda & \cdots & a_4^3\Lambda(1) & \cdots & a_4^3\Lambda(2)
 \end{array}$$

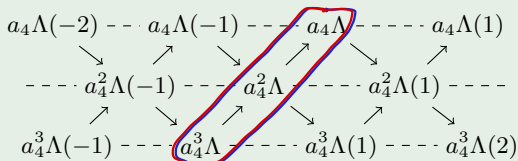
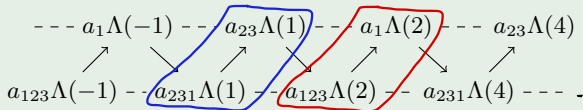
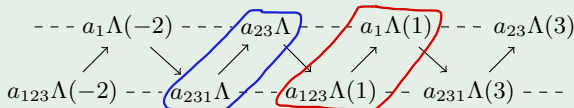
Example (continued)

- The Auslander-Reiten quiver of $\text{Gproj}^{\mathbb{Z}}\Lambda$ is given as follows:



$$\mathcal{P}(\Omega) = \{a_{123}, a_4\}$$

$$\mathcal{P}(\Omega) = \{a_{231}, a_4\}$$



Theorem (Honma-U 2024)

- 1 $T = \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \leq i < l(c)} T_c(i)$ is a tilting object of $\underline{\text{Gproj}}^{\mathbb{Z}} \Lambda$, namely,
(i) $\underline{\text{Hom}}_{\Lambda}^{\mathbb{Z}}(T, \Sigma^i T) = 0$ for $i \neq 0$; (ii) $\text{thick } T = \underline{\text{Gproj}}^{\mathbb{Z}} \Lambda$
- 2 $\underline{\text{End}}_{\Lambda}^{\mathbb{Z}} T \cong \prod_{c \in \mathcal{C}(\Lambda)} (K\mathbb{A}_c)^{(l(c))}$, where $\mathbb{A}_c : 1 \rightarrow 2 \rightarrow \cdots \rightarrow |\mathbb{P}_{\Lambda}(c)|$
- 3 $\underline{\text{Gproj}}^{\mathbb{Z}} \Lambda \cong \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^b(\text{mod } K\mathbb{A}_c)^{(l(c))}$ as triangulated categories

Remark The theorem explicitly describes the graded singularity category $\mathcal{D}_{\text{sg}}(\text{mod}^{\mathbb{Z}} \Lambda)$ of Λ when Λ is Iwanaga-Gorenstein and in particular improves a result of Lu-Zhu (2021) for Iwanaga-Gorenstein monomial algebras

Example

Let $\Lambda = KQ/I$ be defined as earlier. Fix $\mathcal{C}(\Lambda) = \{a_{123}, a_4\}$

- $\underline{\text{Gproj}}^{\mathbb{Z}} \Lambda \cong \mathcal{D}^b(\text{mod } K\mathbb{A}_{a_{123}})^{(3)} \times \mathcal{D}^b(\text{mod } K\mathbb{A}_{a_4})$, where

$$\mathbb{A}_{a_{123}} : 1 \rightarrow 2, \quad \mathbb{A}_{a_4} : 1 \rightarrow 2 \rightarrow 3$$

3. Stable categories of Gorenstien-projective modules

Proposition (LZ 2021)

The forgetful functor $F : \text{mod}^{\mathbb{Z}}\Lambda \rightarrow \text{mod}\Lambda$ induces a G -covering

$$\tilde{F}_G : \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \rightarrow \underline{\text{Gproj}}\Lambda$$

in the sense of Asashiba (2011), where G is the cyclic group generated by the automorphism $(1) : \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \rightarrow \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda$

- Thanks to Asashiba (2011), we obtain an equivalence $H : \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda/(1) \xrightarrow{\sim} \underline{\text{Gproj}}\Lambda$ that makes the following diagram commute

$$\begin{array}{ccc} \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda & \xrightarrow{\tilde{F}_G} & \underline{\text{Gproj}}\Lambda \\ & \searrow P & \nearrow H \\ & & \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda/(1) \end{array}$$

- It follows from Section 2 that

$$\underline{\text{Gproj}}^{\mathbb{Z}}\Lambda = \prod_{c \in \mathcal{C}(\Lambda)} \prod_{0 \leq i < l(c)} \text{thick } T_c(i) \quad \text{with} \quad \text{thick } T_c(i) \cong \mathcal{D}^b(\text{mod } K\mathbb{A}_c)$$

- $P(\text{thick } T_c(i)) = P((\text{thick } T_c)(i)) = P(\text{thick } T_c)$ for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$
- $\underline{\text{Gproj}}\Lambda \cong \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda/(1) = \prod_{c \in \mathcal{C}(\Lambda)} P(\text{thick } T_c)$, where $P(\text{thick } T_c) = \text{thick } P(T_c)$

Lemma

- 1 thick $T_c(i) = \text{thick } T_c(j)$ in $\underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \iff i \equiv j \pmod{l(c)}$ for $c \in \mathcal{C}(\Lambda)$ and $i, j \in \mathbb{Z}$
- 2 For $c \in \mathcal{C}(\Lambda)$, the restriction of $P : \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda \rightarrow \underline{\text{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ to $\text{thick } T_c$ induces a G_c -covering

$$P_c : \text{thick } T_c \rightarrow P(\text{thick } T_c)$$

where G_c is the cyclic group generated by the induced automorphism $(l(c)) : \text{thick } T_c \rightarrow \text{thick } T_c$

- 3 For $c \in \mathcal{C}(\Lambda)$, $P(\text{thick } T_c) \cong \text{thick } T_c/(l(c)) \cong \mathcal{D}^b(\text{mod } K\mathbb{A}_c)/\tau^{|c|}$, where τ is the Auslander-Reiten translation for $\mathcal{D}^b(\text{mod } K\mathbb{A}_c)$

Theorem (Honma-U 2024)

$$\begin{aligned} \underline{\text{Gproj}} \Lambda &\cong \prod_{c \in \mathcal{C}(\Lambda)} P(\text{thick } T_c) \\ &\cong \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^b(\text{mod } K\mathbb{A}_c)/\tau^{|c|} \\ &\cong \prod_{c \in \mathcal{C}(\Lambda)} \underline{\text{mod}} K \left(1 \begin{array}{c} \xrightarrow{\quad} 2 \xrightarrow{\quad} \cdots \xrightarrow{\quad} |c| \\ \xleftarrow{\quad} \end{array} \right) / R^{|\mathbb{P}_\Lambda(c)|+1} \end{aligned}$$

Remark The theorem explicitly describes the singularity categories $\mathcal{D}_{\text{sg}}(\text{mod } \Lambda)$ of Iwanaga-Gorenstein monomial algebras Λ . Moreover, it recovers results of Ringel (2013), Chen-Shen-Zhou (2018), and Lu-Zhu (2021).

Example

Let $\Lambda = KQ/I$ be defined as earlier:

$$\begin{array}{ccccc}
 1 & \xrightarrow{a_1} & 2 & \xrightarrow{b_2} & 4 & \begin{array}{c} \circlearrowright \\ a_4 \end{array} \\
 & \swarrow a_3 & \downarrow a_2 & & \\
 & & 3 & &
 \end{array}
 \quad a_{1231} = a_{23123} = a_4^4 = 0$$

- Fix $\mathcal{C}(\Lambda) = \{a_{123}, a_4\}$. We know $|a_{123}| = 2$, $|a_4| = 1$, $|\mathbb{P}_\Lambda(a_{123})| = 2$, $|\mathbb{P}_\Lambda(a_4)| = 3$
- $\underline{\text{Gproj}} \Lambda \cong \underline{\text{mod}} K(1 \overleftarrow{\rightarrow} 2)/R^3 \times \underline{\text{mod}} K[x]/(x^4)$
- The Auslander-Reiten quiver of $\underline{\text{Gproj}} \Lambda$ is given as follows:

