The stable category of Gorenstein-projective modules over a monomial algebra

Satoshi Usui (Tokyo Metropolitan College of Industrial Technology)
Takahiro Honma (National Institute of Technology (KOSEN), Yuge College)

September 17, 2024

The 56th Symposium on Ring Theory and Representation Theory

Table of contents

Notation

- Λ : a finite dimensional algebra over a field K
- $\mathrm{mod}\,\Lambda$: the category of (finitely generated right) Λ-modules
- $\mathrm{proj}\,\Lambda$: the category of projective Λ-modules

1. Motivation

Definition (Enochs-Jenda 1995)

M_Λ : Gorenstein-projective (GP) $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$ an acyclic complex of projective Λ-modules

$$P^\bullet : \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \xrightarrow{d^1} P^2 \to \cdots$$

s.t. (i) $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is exact; and (ii) $M \cong \mathrm{Ker}\, d^0$ as Λ-modules

- $P^\bullet$: a complete resolution of M
- GP modules are called modules of Gorenstein dimension zero (Auslander-Bridger 1969), totally reflexive modules (Avramov-Martsinkovsky 2002), and maximal Cohen-Macaulay modules (Buchweitz 1989)
- $\mathrm{Gproj}\,\Lambda$: the category of GP Λ-modules $\Longrightarrow$ $\mathrm{proj}\,\Lambda \subseteq \mathrm{Gproj}\,\Lambda \subseteq \mathrm{mod}\,\Lambda$
- $\mathrm{Gproj}\,\Lambda = \mathrm{mod}\,\Lambda \iff \Lambda$: self-injective

Definition

Λ : CM-free $\overset{\text{def}}{\Longleftrightarrow}$ $\mathrm{proj}\,\Lambda = \mathrm{Gproj}\,\Lambda$

- $\mathrm{gl.dim}\,\Lambda < \infty \Longrightarrow \Lambda$: CM-free

- From now on, assume Λ is monomial (i.e. $\Lambda = KQ/I$, where I is generated by paths)

$\Longrightarrow$ Λ is CM-finite (i.e. $\# \operatorname{ind} \operatorname{Gproj} \Lambda < \infty$) since any indecomposable non-projective GP Λ-module is of the form $p\Lambda$ for some non-zero non-trivial path p

Definition (Chen-Shen-Zhou 2018)

- A pair (p, q) of non-zero paths in Λ is perfect if the following are satisfied:
  1. p and q are both non-trivial with $t(p) = s(q)$ and satisfy $pq = 0$ in Λ
  2. If $pq' = 0$ for a non-zero path q' with $t(p) = s(q')$, then $q' = qq''$ for some path q''
  3. If $p'q = 0$ for a non-zero path p' with $t(p') = s(q)$, then $p' = p''p$ for some path p''
- $(p_1, \ldots, p_n, p_{n+1} = p_1)$: a perfect path sequence if (p_i, p_{i+1}) is perfect for $1 \leq i \leq n$
- A path in a perfect path sequence is called a perfect path

- $\mathbb{P}_\Lambda$: the set of perfect paths

Theorem (CSZ 2018)

$$\begin{array}{ccc} \mathbb{P}_\Lambda & \overset{1:1}{\longleftrightarrow} & \{\text{indecomposable non-projective GP } \Lambda\text{-modules}\}/\cong \\ p & \longmapsto & p\Lambda \end{array}$$

- $\mathbb{P}_\Lambda$ is empty $\iff$ Λ is CM-free

<u>Remark</u> Perfect path sequences give rise to complete resolutions

Definition (Chen-Shen-Zhou 2018)

- A pair (p, q) of non-zero paths in Λ is perfect if the following are satisfied:
  1. p and q are both non-trivial with $t(p) = s(q)$ and satisfy $pq = 0$ in Λ
  2. If $pq' = 0$ for a non-zero path q' with $t(p) = s(q')$, then $q' = qq''$ for some path q''
  3. If $p'q = 0$ for a non-zero path p' with $t(p') = s(q)$, then $p' = p''p$ for some path p''
- $(p_1, \ldots, p_n, p_{n+1} = p_1)$: a perfect path sequence if (p_i, p_{i+1}) is perfect for $1 \leq i \leq n$
- A path in a perfect path sequence is called a perfect path

Example

Consider the monomial algebra $\Lambda = K \left(\begin{tikzcd} 1 \arrow[loop right, "x"] \end{tikzcd} \right) \Big/ (x^5)$

- $(x,\ x^4,\ x)$, $(x^2,\ x^3,\ x^2)$: the minimal perfect path sequences
- $\mathbb{P}_\Lambda = \{x, x^2, x^3, x^4\}$, hence $\operatorname{ind} \operatorname{Gproj} \Lambda = \{x\Lambda, x^2\Lambda, x^3\Lambda, x^4\Lambda\} \cup \operatorname{ind} \operatorname{proj} \Lambda$
- For example, $(x,\ x^4,\ x)$ induces the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & x\Lambda \arrow[r] & \Lambda \arrow[rr] \arrow[dr] & & \Lambda \arrow[r] & x\Lambda \arrow[r] & 0 \\ & & & x^4\Lambda \arrow[ur] & & & \end{tikzcd}$$

• The stable category $\underline{\mathrm{Gproj}}\,\Lambda = \mathrm{Gproj}\,\Lambda / \mathrm{proj}\,\Lambda$ of $\mathrm{Gproj}\,\Lambda$ carries a structure of a triangulated category

Theorem (CSZ 2018)

TFAE

1. $\underline{\mathrm{Gproj}}\,\Lambda$ is a semisimple triangulated category
2. $\exists$ no overlap in Λ
3. $\underline{\mathrm{Gproj}}\,\Lambda \cong \prod_{i=1}^{n} (\mathrm{mod}\,k^{n_i}, \sigma^*)$, where the automorphism $\sigma^* : \mathrm{mod}\,k^{n_i} \to \mathrm{mod}\,k^{n_i}$ is induced by $\sigma : k^{n_i} \to k^{n_i} \in \mathrm{Aut}\,\Lambda$ given by $\sigma(\lambda_1, \lambda_2, \ldots, \lambda_{n_i}) = (\lambda_2, \ldots, \lambda_{n_i}, \lambda_1)$

Remark $\exists$ no overlap in Λ $\iff$ there exists no non-trivial morphism in $\underline{\mathrm{Gproj}}\,\Lambda$

• Ringel (2013) and Lu-Zhu (2021) determined $\underline{\mathrm{Gproj}}\,\Lambda$ for Nakayama algebras and 1-Iwanaga-Gorenstein monomial algebras, respectively

Recall Λ : (d-)Iwanaga-Gorenstein $\overset{\mathrm{def}}{\iff}$ $\mathrm{id}_\Lambda \Lambda,\ \mathrm{id}\,\Lambda_\Lambda \leq d < \infty$

• In any cases, $\underline{\mathrm{Gproj}}\,\Lambda \cong \underline{\mathrm{mod}}\,\Gamma$ for some self-injective Nakayama algebra Γ

Our aim is

to describe $\underline{\mathrm{Gproj}}\,\Lambda$ for more general monomial algebras Λ

Remark Many authors such as Chen-Geng-Lu (2015), Lu (2016, 2019), Enomoto (2018) and Minamoto-Yamaura (2020) describe $\underline{\mathrm{Gproj}}\,\Lambda$ for specific classes of non-monomial (Iwanaga-Gorenstein) algebras Λ

Theorem (Buchweitz 1986)

If Λ is an Iwanaga-Gorenstein algebra, then $\underline{\mathrm{Gproj}}\,\Lambda$ is triangle equivalent to the singularity category $\mathcal{D}_{\mathrm{sg}}(\mathrm{mod}\,\Lambda) := \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)/\mathcal{K}^{\mathrm{b}}(\mathrm{proj}\,\Lambda)$

Definition

The underlying cycle c_p associated with $p \in \mathbb{P}_\Lambda$ is the shortest cycle c s.t. $p_1 \cdots p_n = c^l$ for some $l > 0$, where $(p = p_1, \ldots, p_n, p_{n+1} = p_1)$ is a perfect path sequence

- $\mathcal{C}(\Lambda)$: the set of equivalence classes (w.r.t. cyclic permutation) of underlying cycles

Definition

For p and $q \in \mathbb{P}_\Lambda$, we write $p \preceq q$ if $q = pr$ for some path r

- $(\mathbb{P}_\Lambda, \preceq)$ is a poset
- The Hasse quiver $H(\mathbb{P}_\Lambda, \preceq)$ is a disjoint union of linear quivers

Definition

$p \in \mathbb{P}_\Lambda$: co-elementary $\overset{\text{def}}{\Longleftrightarrow}$ p is a sink in $H(\mathbb{P}_\Lambda, \preceq)$

- $\mathbb{E}^{\mathrm{co}}_\Lambda$: the set of co-elementary paths

Example

Let $\Lambda = KQ/I$ be the monomial algebra given by

$$\begin{tikzcd} 1 \arrow[r, "a_1"] & 2 \arrow[r, "b_2"] \arrow[d, "a_2"] & 4 \arrow[loop right, "a_4"] \\ & 3 \arrow[ul, "a_3"] & \end{tikzcd} \qquad a_{1231} = a_{23123} = a_4^4 = 0$$

- The following are the minimal perfect path sequences

$$(a_1,\ a_{231},\ a_{23},\ a_{123},\ a_1), \quad (a_4,\ a_4^3,\ a_4), \quad (a_4^2,\ a_4^2)$$

- $\mathbb{P}_\Lambda = \{a_1,\, a_{231},\, a_{23},\, a_{123},\, a_4,\, a_4^2,\, a_4^3\}$

- $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$, where $a_{123} = a_{231}$

- $H(\mathbb{P}_\Lambda, \preceq)$: $\quad a_{123} \longrightarrow a_1 \qquad a_{231} \longrightarrow a_{23} \qquad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$

- $\mathbb{E}_\Lambda^{\mathrm{co}} = \{a_1, a_{23}, a_4\}$

Proposition-Definition

For $c \in \mathcal{C}(\Lambda)$, $\exists!\, r_1, \ldots, r_n \in \mathbb{E}_\Lambda^{\mathrm{co}}$ s.t. $c = r_1 \cdots r_n$. We denote $|c| := n$.

- Consider $\Lambda = KQ/I$ as a positively graded algebra by defining $\deg a = 1$ for $a \in Q_1$
- $\mathrm{mod}^{\mathbb{Z}}\Lambda$: the category of graded Λ-modules

Recall For $M, N \in \mathrm{mod}^{\mathbb{Z}}\Lambda$, $\mathrm{Hom}^{\mathbb{Z}}_{\Lambda}(M,N) := \{f \in \mathrm{Hom}_{\Lambda}(M,N) \mid f(M_i) \subseteq N_i$ for $i\}$

- $\mathrm{proj}^{\mathbb{Z}}\Lambda$: the category of graded projective Λ-modules
- $\mathrm{Gproj}^{\mathbb{Z}}\Lambda$: the category of graded GP Λ-modules
- $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda = \mathrm{Gproj}^{\mathbb{Z}}\Lambda/\mathrm{proj}^{\mathbb{Z}}\Lambda$: the stable category of $\mathrm{Gproj}^{\mathbb{Z}}\Lambda$
- Lu-Zhu (2021) observed that $\mathrm{ind}\,\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda = \{p\Lambda(i) \mid p \in \mathbb{P}_{\Lambda}, i \in \mathbb{Z}\}$

Theorem (LZ 2021)

If Λ is Iwanaga-Gorenstein, then $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\,H)$ for some hereditary algebra H of finite representation type

Remark They use the triangle equivalence

$$\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \;\cong\; \mathcal{D}_{\mathrm{sg}}(\mathrm{mod}^{\mathbb{Z}}\Lambda) := \mathcal{D}^{\mathrm{b}}(\mathrm{mod}^{\mathbb{Z}}\Lambda)/\mathcal{K}^{\mathrm{b}}(\mathrm{proj}^{\mathbb{Z}}\Lambda)$$

- For $c = r_1 \cdots r_n \in \mathcal{C}(\Lambda)$ with $r_i \in \mathbb{E}_\Lambda^{\mathrm{co}}$, we define

$$\mathbb{P}_\Lambda(c) := \{p \in \mathbb{P}_\Lambda \mid r_1 \preceq p\} \quad \text{and} \quad T_c := \bigoplus_{p \in \mathbb{P}_\Lambda(c)} p\Lambda$$

- Define $T := \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \leq i < l(c)} T_c(i) \in \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$

Example

Let $\Lambda = KQ/I$ be defined as earlier:

$$\begin{tikzcd} 1 \arrow[r, "a_1"] & 2 \arrow[r, "b_2"] \arrow[d, "a_2"] & 4 \arrow[loop right, "a_4"] \\ & 3 \arrow[ul, "a_3"] & \end{tikzcd} \qquad a_{1231} = a_{23123} = a_4^4 = 0$$

- $H(\mathbb{P}_\Lambda, \preceq):$ $\quad a_{123} \longrightarrow a_1 \qquad a_{231} \longrightarrow a_{23} \qquad a_4^3 \longrightarrow a_4^2 \longrightarrow a_4$

- $\mathcal{C}(\Lambda) = \{a_{123},\ a_4\}$

$\Longrightarrow \ T = a_1\Lambda \oplus a_{123}\Lambda \oplus a_1\Lambda(1) \oplus a_{123}\Lambda(1) \oplus a_1\Lambda(2) \oplus a_{123}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$

- $\mathcal{C}(\Lambda) = \{a_{231},\ a_4\}$

$\Longrightarrow \ T = a_{23}\Lambda \oplus a_{231}\Lambda \oplus a_{23}\Lambda(1) \oplus a_{231}\Lambda(1) \oplus a_{23}\Lambda(2) \oplus a_{231}\Lambda(2) \oplus a_4\Lambda \oplus a_4^2\Lambda \oplus a_4^3\Lambda$

Example (continued)

- The Auslander-Reiten quiver of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ is given as follows:

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[r, dash, dashed] & a_1\Lambda(-3) \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda(-1) \arrow[dr] \arrow[rr, dash, dashed] & & a_1\Lambda \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda(2) & \\
a_{123}\Lambda(-3) \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda(-1) \arrow[ur] \arrow[rr, dash, dashed] & & a_{123}\Lambda \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda(2) \arrow[ur] \arrow[rr, dash, dashed] & & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[r, dash, dashed] & a_1\Lambda(-2) \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda \arrow[dr] \arrow[rr, dash, dashed] & & a_1\Lambda(1) \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda(3) & \\
a_{123}\Lambda(-2) \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda \arrow[ur] \arrow[rr, dash, dashed] & & a_{123}\Lambda(1) \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda(3) \arrow[ur] \arrow[rr, dash, dashed] & & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
\cdots \arrow[r, dash, dashed] & a_1\Lambda(-1) \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda(1) \arrow[dr] \arrow[rr, dash, dashed] & & a_1\Lambda(2) \arrow[dr] \arrow[rr, dash, dashed] & & a_{23}\Lambda(4) & \\
a_{123}\Lambda(-1) \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda(1) \arrow[ur] \arrow[rr, dash, dashed] & & a_{123}\Lambda(2) \arrow[ur] \arrow[rr, dash, dashed] & & a_{231}\Lambda(4) \arrow[ur] \arrow[rr, dash, dashed] & & \cdots
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
& a_4\Lambda(-2) \arrow[dr] \arrow[rr, dash, dashed] & & a_4\Lambda(-1) \arrow[dr] \arrow[rr, dash, dashed] & & a_4\Lambda \arrow[dr] \arrow[rr, dash, dashed] & & a_4\Lambda(1) & \\
\cdots \arrow[rr, dash, dashed] & & a_4^2\Lambda(-1) \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & a_4^2\Lambda \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & a_4^2\Lambda(1) \arrow[ur] \arrow[dr] \arrow[rr, dash, dashed] & & \cdots \\
& a_4^3\Lambda(-1) \arrow[ur] \arrow[rr, dash, dashed] & & a_4^3\Lambda \arrow[ur] \arrow[rr, dash, dashed] & & a_4^3\Lambda(1) \arrow[ur] \arrow[rr, dash, dashed] & & a_4^3\Lambda(2) &
\end{tikzcd}
$$

Example (continued)

- The Auslander-Reiten quiver of $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda$ is given as follows:

$$
\begin{tikzcd}[column sep=small]
 & a_1\Lambda(-3) \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda(-1) \arrow[dr] \arrow[rr, dashed, no head] & & a_1\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda(2) \\
a_{123}\Lambda(-3) \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda(-1) \arrow[ur] \arrow[rr, dashed, no head] & & a_{123}\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda(2) \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & a_1\Lambda(-2) \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_1\Lambda(1) \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda(3) \\
a_{123}\Lambda(-2) \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_{123}\Lambda(1) \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda(3) \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}[column sep=small]
 & a_1\Lambda(-1) \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda(1) \arrow[dr] \arrow[rr, dashed, no head] & & a_1\Lambda(2) \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda(4) \\
a_{123}\Lambda(-1) \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda(1) \arrow[ur] \arrow[rr, dashed, no head] & & a_{123}\Lambda(2) \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda(4) \arrow[ur] &
\end{tikzcd}
.
$$

$$
\begin{tikzcd}[column sep=small]
a_4\Lambda(-2) \arrow[dr] \arrow[rr, dashed, no head] & & a_4\Lambda(-1) \arrow[dr] \arrow[rr, dashed, no head] & & a_4\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_4\Lambda(1) \\
 & a_4^2\Lambda(-1) \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & a_4^2\Lambda \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & a_4^2\Lambda(1) \arrow[ur] \arrow[dr] & \\
a_4^3\Lambda(-1) \arrow[ur] \arrow[rr, dashed, no head] & & a_4^3\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_4^3\Lambda(1) \arrow[ur] \arrow[rr, dashed, no head] & & a_4^3\Lambda(2)
\end{tikzcd}
$$

$e(\Lambda) = \{a_{123}, a_4\}$

$e(\Lambda) = \{a_{231}, a_4\}$

Theorem (Honma-U 2024)

1. $T = \bigoplus_{c \in \mathcal{C}(\Lambda)} \bigoplus_{0 \leq i < l(c)} T_c(i)$ is a tilting object of $\underline{\mathrm{Gproj}}^{\mathbb{Z}} \Lambda$, namely,

$$\text{(i) } \underline{\mathrm{Hom}}_{\Lambda}^{\mathbb{Z}}(T, \Sigma^i T) = 0 \text{ for } i \neq 0; \qquad \text{(ii) } \mathrm{thick}\, T = \underline{\mathrm{Gproj}}^{\mathbb{Z}} \Lambda$$

2. $\underline{\mathrm{End}}_{\Lambda}^{\mathbb{Z}} T \cong \prod_{c \in \mathcal{C}(\Lambda)} (K\mathbb{A}_c)^{(l(c))}$, where $\mathbb{A}_c : 1 \to 2 \to \cdots \to |\mathbb{P}_{\Lambda}(c)|$

3. $\underline{\mathrm{Gproj}}^{\mathbb{Z}} \Lambda \cong \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)^{(l(c))}$ as triangulated categories

Remark The theorem explicitly describes the graded singularity category $\mathcal{D}_{\mathrm{sg}}(\mathrm{mod}^{\mathbb{Z}} \Lambda)$ of Λ when Λ is Iwanaga-Gorenstein and in particular improves a result of Lu-Zhu (2021) for Iwanaga-Gorenstein monomial algebras

Example

Let $\Lambda = KQ/I$ be defined as earlier. Fix $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$

- $\underline{\mathrm{Gproj}}^{\mathbb{Z}} \Lambda \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_{a_{123}})^{(3)} \times \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_{a_4})$, where

$$\mathbb{A}_{a_{123}} : 1 \to 2, \qquad \mathbb{A}_{a_4} : 1 \to 2 \to 3$$

Proposition (LZ 2021)

The forgetful functor $F : \operatorname{mod}^{\mathbb{Z}}\Lambda \to \operatorname{mod}\Lambda$ induces a G-covering

$$\tilde{F}_G : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\operatorname{Gproj}}\,\Lambda$$

in the sense of Asashiba (2011), where G is the cyclic group generated by the automorphism $(1) : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda$

- Thanks to Asashiba (2011), we obtain an equivalence $H : \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda/(1) \xrightarrow{\sim} \underline{\operatorname{Gproj}}\,\Lambda$ that makes the following diagram commute

$$\begin{tikzcd} \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda \arrow[rr, "\tilde{F}_G"] \arrow[dr, "P"'] & & \underline{\operatorname{Gproj}}\,\Lambda \\ & \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda/(1) \arrow[ur, "\sim", "H"'] & \end{tikzcd}$$

- It follows from Section 2 that

$$\underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda = \prod_{c\in\mathcal{C}(\Lambda)} \prod_{0\le i<l(c)} \operatorname{thick} T_c(i) \quad \text{with} \quad \operatorname{thick} T_c(i) \cong \mathcal{D}^{\mathrm{b}}(\operatorname{mod} K\mathbb{A}_c)$$

- $P(\operatorname{thick} T_c(i)) = P((\operatorname{thick} T_c)(i)) = P(\operatorname{thick} T_c)$ for $c \in \mathcal{C}(\Lambda)$ and $i \in \mathbb{Z}$
- $\underline{\operatorname{Gproj}}\,\Lambda \cong \underline{\operatorname{Gproj}}^{\mathbb{Z}}\Lambda/(1) = \prod_{c\in\mathcal{C}(\Lambda)} P(\operatorname{thick} T_c)$, where $P(\operatorname{thick} T_c) = \operatorname{thick} P(T_c)$

Lemma

1. $\mathrm{thick}\, T_c(i) = \mathrm{thick}\, T_c(j)$ in $\underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \iff i \equiv j \pmod{l(c)}$ for $c \in \mathcal{C}(\Lambda)$ and $i, j \in \mathbb{Z}$
2. For $c \in \mathcal{C}(\Lambda)$, the restriction of $P : \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda \to \underline{\mathrm{Gproj}}^{\mathbb{Z}}\Lambda/(1)$ to $\mathrm{thick}\, T_c$ induces a G_c-covering
$$P_c : \mathrm{thick}\, T_c \to P(\mathrm{thick}\, T_c)$$
where G_c is the cyclic group generated by the induced automorphism $(l(c)) : \mathrm{thick}\, T_c \to \mathrm{thick}\, T_c$
3. For $c \in \mathcal{C}(\Lambda)$, $P(\mathrm{thick}\, T_c) \cong \mathrm{thick}\, T_c/(l(c)) \cong \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|}$, where τ is the Auslander-Reiten translation for $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)$

Threoem (Honma-U 2024)

$$\begin{aligned}
\underline{\mathrm{Gproj}}\,\Lambda &\cong \prod_{c \in \mathcal{C}(\Lambda)} P(\mathrm{thick}\, T_c) \\
&\cong \prod_{c \in \mathcal{C}(\Lambda)} \mathcal{D}^{\mathrm{b}}(\mathrm{mod}\, K\mathbb{A}_c)/\tau^{|c|} \\
&\cong \prod_{c \in \mathcal{C}(\Lambda)} \underline{\mathrm{mod}}\, K \left(\begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & {|c|} \arrow[lll, bend left] \end{tikzcd} \right) \Big/ R^{|\mathbb{P}_\Lambda(c)|+1}
\end{aligned}$$

Remark The theorem explicitly describes the singularity categories $\mathcal{D}_{\mathrm{sg}}(\operatorname{mod}\Lambda)$ of Iwanaga-Gorenstein monomial algebras Λ. Moreover, it recovers results of Ringel (2013), Chen-Shen-Zhou (2018), and Lu-Zhu (2021).

Example

Let $\Lambda = KQ/I$ be defined as earlier:

$$
\begin{tikzcd}
1 \arrow[r, "a_1"] & 2 \arrow[r, "b_2"] \arrow[d, "a_2"] & 4 \arrow[loop right, "a_4"] \\
& 3 \arrow[ul, "a_3"] &
\end{tikzcd}
\qquad a_{1231} = a_{23123} = a_4^4 = 0
$$

- Fix $\mathcal{C}(\Lambda) = \{a_{123},\, a_4\}$. We know $|a_{123}| = 2,\ |a_4| = 1,\ |\mathbb{P}_\Lambda(a_{123})| = 2,\ |\mathbb{P}_\Lambda(a_4)| = 3$
- $\underline{\mathrm{Gproj}}\,\Lambda \cong \underline{\mathrm{mod}}\,K\big(1 \rightleftarrows 2\big)/R^3 \times \underline{\mathrm{mod}}\,K[x]/(x^4)$
- The Auslander-Reiten quiver of $\underline{\mathrm{Gproj}}\,\Lambda$ is given as follows:

$$
\begin{tikzcd}
& a_1\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_{23}\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_1\Lambda \\
a_{123}\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_{231}\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_{123}\Lambda \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & a_4\Lambda \arrow[dr] \arrow[rr, dashed, no head] & & a_4\Lambda \\
& a_4^2\Lambda \arrow[ur] \arrow[dr] \arrow[rr, dashed, no head] & & a_4^2\Lambda \arrow[ur] & \\
a_4^3\Lambda \arrow[ur] \arrow[rr, dashed, no head] & & a_4^3\Lambda \arrow[ur] & &
\end{tikzcd}
$$