# Foundation of the Representation Theory of Artin Algebras, Using the Gabriel-Roiter Measure.

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## 1. The Setting.

Let $\Lambda$ be an artin algebra (this means that $\Lambda$ is an associative ring with 1, its center is a commutative artinian ring and $\Lambda$ is finitely generated as a module over its center), we always may (and will) assume that $\Lambda$ is connected (this means that the center is a local ring). Let $\operatorname{Mod}\Lambda$ denote the category of all (left) $\Lambda$-modules and $\operatorname{mod}\Lambda$ the full subcategory of all finitely generated modules. Usually, we will deal with finitely generated modules and call them just *modules*, given such a module $M$, we denote by $|M|$ its length (this is the length of any composition series, recall that this is an invariant of the module according to the Jordan-Hölder theorem).

Our interest concerns indecomposable modules: given an arbitrary, not necessarily finitely generated module $M$ and submodules $M_1, M_2$ of $M$, then we write $M = M_1 \oplus M_2$ provided $M_1 \cap M_2 = 0$ and $M_1 + M_2 = M$ and call this a *direct decomposition* of $M$; we say that $M$ is *indecomposable,* provided $M$ is nonzero and the only direct decompositions $M = M_1 \oplus M_2$ are those with $M_1 = 0$ or $M_2 = 0$. Of course, any finitely generated $\Lambda$-module can be written as a finite direct sum of indecomposable modules, and such a decomposition is unique up to isomorphism (according to the Theorem of Krull-Remak-Schmidt); the reason for this uniqueness is the fact that any indecomposable module of finite length has a local endomorphism ring.

The main problem of representation theory is to find invariants for modules and to describe the isomorphism classes of all the indecomposable modules for which such an invariant takes a fixed value. A typical such invariant is the length of a module: the simple modules are those of length 1 (and there is just a finite number of such modules), the information concerning the indecomposable modules of length 2 is stored in the quiver (in case we deal with a finite dimensional algebra over some algebraically closed field) or the "species" of $\Lambda$. Given any invariant $\gamma$, as a first question one may look for values *of finite type:* these are those values $v$ such that there are only finitely many isomorphisms classes of indecomposable modules $M$ with $\gamma(M) = v$. The invariant to be discussed here is the Gabriel-Roiter measure.

The Gabriel-Roiter measure was introduced (under the name "Roiter measure") by Gabriel in [G] in order to clarify the intricate induction scheme used by Roiter [Ro] in his proof of the first Brauer-Thrall conjecture. Gabriel's analysis of Roiter's proof is a quite non-trivial achievement and it merits to add his name to the concept. Indeed, the definition of what we call the Gabriel-Roiter measure seems to be strange on first sight, but as we are going to show it embodies a complete theory. Recall that the first Brauer-Thrall conjecture [Ri3] asserted that an artin algebra of bounded representation type is of finite representation type (here, *bounded representation type* means that there is a bound on the length of the indecomposable representations, and *finite representation type* means that there are only finitely many isomorphism classes of indecomposables). Roiter's proof of this conjecture marks the beginning of the new representation theory of finite dimensional algebras. Despite the fame of the result, the actual paper of Roiter (and also Gabriel's interpretation) was apparently forgotten in the meantime. There was a later proof of the first Brauer-Thrall conjecture by Auslander and it is this proof, or its modification due to Yamagata, which usually is presented. Auslander's proof has the advantage that it works for artinian rings, not only for artin algebras, but the usual references do not even exploit this, but use it as a striking application of the Auslander-Reiten theory for artin algebras (which it is). It is worthwhile to recall the old proof of Roiter and the methods involved. These methods can be used and should be used as a kind of foundation for the representation theory of artin algebras: the Gabriel-Roiter measure seems to be an important first invariant to be studied when dealing with the representations of an artin algebra. One of the reasons that this has not been done may stem from the fact that both Roiter as well as Gabriel work from the beginning only with algebras of bounded representation type (thus with algebras which are shown to be of finite representation type). However, and this will be our main objective, the Gabriel-Roiter measure can be introduced and used for arbitrary artin algebras, and it unfolds its real strength when dealing with algebras of infinite representation type! (Actually, there is a footnote in Gabriel's paper asserting that one may waive the restriction of dealing with bounded representation type, but apparently this was overlooked.)

The main topic to be discussed here will be cogeneration of modules: Recall that given two modules $X, Y$, one says that $X$ is *cogenerated by* $Y$ provided the intersection of the kernels of all maps $X \to Y$ is zero. In case $X$ is of finite length, it is immediate to see that $X$ is cogenerated by $Y$ if and only if $X$ can be embedded into a finite direct sum of copies of $Y$. Cogeneration yields a kind of partial ordering of the isomorphism classes of $\Lambda$-modules. Namely, there is the following observation:

*Assume that $X, Y$ are non-zero modules of finite length such that $X$ is cogenerated by $Y$ and $Y$ is cogenerated by $X$, then there is an indecomposable module $Z$ which is a direct summand of $X$ as well as of $Y$.*

Proof: By assumption, there exist embeddings $f\colon X \to Y^n$ and $g\colon Y \to X^m$ for some natural numbers $n, m$. Obviously, this yields an inclusion map $h\colon X \to X^{nm}$ which factors through $Y^n$. Since for any module $X$ the radical of the endomorphism ring of $X$ annihilates some non-zero element of $X$, we conclude that there is an indecomposable direct summand $X'$ with inclusion $m\colon X' \to X'$ and a projection $p\colon X^{mn} \to X'$ such that the composition $phm\colon X' \to X'$ is invertible. Since this invertible map $phm$ factors through $Y^n$, the module $X'$ occurs as a direct summand of $Y^n$ and therefore of $Y$.

The Gabriel-Roiter measure $\mu$ provides a tool for a better understanding of the cogeneration of modules. It allows to index the isomorphism classes of the $\Lambda$-modules by a totally ordered set (say a set of real numbers with their usual ordering) so that cogenerations are possible only in the given order: *Assume that $X, Y$ are non-zero modules of finite length and without any common indecomposable direct summand. If and $X$ is cogenerated by $Y$, then $\mu(X) < \mu(Y)$.*

For the proofs of the Main Proposition and Theorems 1, 2 and 3, see [R5].

## 2. The Basic Definitions.

Let $\mathbb{N}_1 = \{1, 2, \dots\}$ be the set of natural numbers. Note that we use the symbol $\subset$ to denote proper inclusions. Let $\mathcal{P}(\mathbb{N}_1)$ be the set of all subsets $I \subseteq \mathbb{N}_1$. We consider this set as a **totally ordered** set as follows: If $I, J$ are different subsets of $\mathbb{N}_1$, write $I < J$ provided the smallest element in $(I \setminus J) \cup (J \setminus I)$ belongs to $J$. It is easy to see that $\mathcal{P}(\mathbb{N}_1)$ with this ordering is complete. Also note that $I \subseteq J \subseteq \mathbb{N}_1$ implies that $I \leq J$.

The Gabriel-Roiter measure of a module of finite length will be a finite set of natural numbers. We want to provide a more intuitive understanding of the Gabriel-Roiter measure, in particular of the total ordering as described above. In order to do so, we are going to embed the set $\mathcal{P}_f(\mathbb{N}_1)$ of all finite subsets of $\mathbb{N}_1$ into the ordered set $\mathbb{Q}$ of all rational numbers (in section 5 we will extend this embedding to an embedding of all the possible Gabriel-Roiter measures for arbitrary, not necessarily finitely generated modules over an artin algebra into the ordered set of real numbers).

**Lemma 1.** *The map $r \colon \mathcal{P}_f(\mathbb{N}_1) \to \mathbb{Q}$ given by $r(I) = \sum_{i \in I} \frac{1}{2^i}$ for $I \in \mathcal{P}_f(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0, 1]$ and it preserves and reflects the ordering.*

Proof: The essential consideration is the following: Let $I, J$ belong to $\mathcal{P}_f(\mathbb{N}_1)$ with $I < J$. Then $r(I) = r(I \cap J) + r(I \setminus J)$ and $r(J) = r(I \cap J) + r(J \setminus I)$. Let $a$ be the smallest element in $J \setminus I$. Then $r(J \setminus I) \geq \frac{1}{2^a} = \sum_{i > a} \frac{1}{2^i} > r(I \setminus J)$, since $I \setminus J$ is a proper subset of $\{i \in \mathbb{N}_1 \mid i > a\}$.

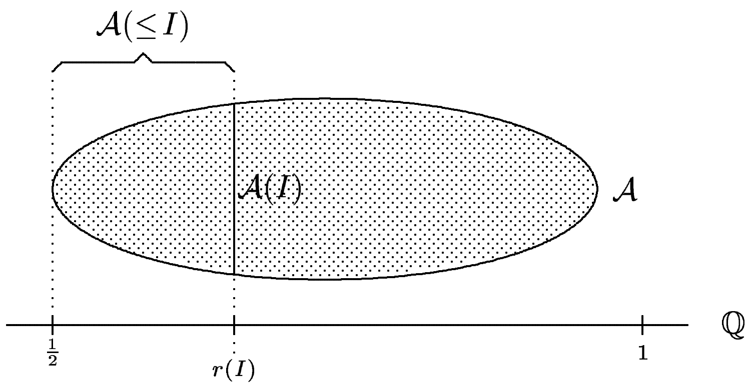
For a (not necessarily finitely generated) $\Lambda$-module $M$, let $\mu(M)$ be the supremum of the sets $\{|M_1|, \dots, |M_t|\}$ in the complete totally ordered set $(\mathcal{P}(\mathbb{N}_1), \leq)$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of indecomposable submodules of $M$. We call $\mu(M)$ the *Gabriel-Roiter measure* of $M$. Note that the Gabriel-Roiter measure of a module $M$ only depends on its submodule lattice: if $M$ and $N$ are modules with isomorphic submodule lattices, then $\mu(M) = \mu(N)$.

**Examples.** Let $M$ be an indecomposable module of length $t$.

- $\mu(M) = \{1\}$ iff $M$ is simple (thus $t = 1$).
- $\mu(M) = \{1, 2\}$ iff $M$ is indecomposable and $t = 2$.
- $\mu(M) = \{1, 2, \dots, t\}$ iff $M$ is uniform (i.e. its socle is simple).
- $\mu(M) = \{1, t\}$ iff $M$ is local and has Loewy length at most 2.

We will use the Gabriel-Roiter measure $\mu$ (or the composition $r\mu$) in order to visualize the category $\operatorname{mod} \Lambda$. As abbreviation, let us write $\mathcal{A} = \operatorname{mod} \Lambda$. For any finite subset $I \subset \mathbb{N}_1$, we denote by $\mathcal{A}(I)$ the class of indecomposable $\Lambda$-modules $M$

with $\mu(M) = I$, and we say that $I$ is a *Gabriel-Roiter measure for* $\Lambda$ provided $\mathcal{A}(I)$ is non-empty. Similarly, let $\mathcal{A}(\leq I)$ be the class of indecomposable $\Lambda$-modules $M$ with $\mu(M) \leq I$.




If $M$ is an indecomposable $\Lambda$-module of finite length, we call any filtration

$$M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M$$

with $\mu(M) = \{|M_1|, |M_2|, \ldots, |M_{t-1}|, |M_t|\}$ a *Gabriel-Roiter filtration* of $M$; if $M$ is of length at least 2 (thus $t \geq 2$) the module $M_{t-1}$ will be said to be a *Gabriel-Roiter submodule* of $M$. Thus a Gabriel-Roiter filtration exhibits an iterated sequence of Gabriel-Roiter submodules (in section 5, we will consider also Gabriel-Roiter filtrations of infinitely generated modules, again using iterated sequences of Gabriel-Roiter submodules). Given a proper inclusion $X \subset Y$ of indecomposable finite length modules, then $X$ is a Gabriel-Roiter submodule of $Y$ iff $\mu(Y) = \mu(X) \cup \{|Y|\}$. In particular, if $X$ is a Gabriel-Roiter submodule of $Y$, then for every monomorphism $f\colon X \to Y$, also $f(X)$ is a Gabriel-Roiter submodule of $Y$.

Gabriel-Roiter submodules of a given indecomposable module are usually not unique, not even unique up to isomorphism (all have however the same length). For example, for the Kronecker quiver, all the indecomposables of length 2 are Gabriel-Roiter submodules of the indecomposable injective module of length 3.

## 3. The Cogeneration Property.

**Main Property (Gabriel).** *Let $X, Y_1, \ldots, Y_t$ be indecomposable $\Lambda$-modules of finite length and assume that there is a monomorphism $f\colon X \to \bigoplus_{i=1}^t Y_i$.*
(a) *Then $\mu(X) \leq \max \mu(Y_i)$.*
(b) *If $\max \mu(Y_i)$ starts with $\mu(X)$, then there is some $j$ such that $\pi_j f$ is injective, where $\pi_j\colon \bigoplus_i Y_i \to Y_j$ is the canonical projection.*

Note that (b) immediately implies:
(b′) *If $\mu(X) = \max \mu(Y_i)$, then $f$ splits.*

The assertions (a) and (b′) have been formulated and proven by Gabriel in [G] using the additional assumption that $\Lambda$ is of bounded representation type.

We conclude: $\operatorname{add}\mathcal{A}(\leq I)$ *is closed under submodules and any monomorphism* $f\colon X \to Y$ *with* $X$ *in* $\mathcal{A}(I)$ *and* $Y$ *in* $\operatorname{add}\mathcal{A}(\leq I)$ *splits* (if $\mathcal{X}$ is a class of indecomposable $\Lambda$-modules, we denote by $\operatorname{add}\mathcal{X}$ the class of all finite direct sums of modules in $\mathcal{X}$). The latter assertion may be reformulated as follows: the modules in $\operatorname{add}\mathcal{A}(I)$ are "relative injective" inside $\operatorname{add}\mathcal{A}(\leq I)$.

**Corollary 1.** *If $M_1, \dots, M_t$ are (not necessarily finitely generated) indecomposable $\Lambda$-modules, then $\mu(\bigoplus M_i) = \max \mu(M_i)$.*

Proof: Since $M_i$ is a submodule of $M = \bigoplus M_i$, we have $\max \mu(M_i) \leq \mu(\bigoplus M_i)$. Conversely, $\mu(M)$ is the supremum of $\mu(M')$, where $M'$ is a finitely generated indecomposable submodule of $M$, thus we have to show $\mu(M') \leq \max \mu(M_i)$. Now $M' \subseteq \bigoplus M_i'$, where $M_i'$ is a finitely generated submodule of $M_i$. We can write $M_i' = \bigoplus_j M_{ij}$ with indecomposable modules $M_{ij}$. Note that $M_{ij}$ is a submodule of $M_i$, thus $\mu(M_{ij}) \leq \mu(M_i)$. According to part (a) of Main Property, we get $\mu(M') \leq \max_{ij} \mu(M_{ij}) \leq \max_i M_i$, this concludes the proof.

**Corollary 2.** *Let $M$ be an indecomposable module and $N$ a Gabriel-Roiter submodule of $M$. Then, for any proper submodule $N'$ of $M$ containing $N$, the embedding $N \subseteq N'$ splits.*

Proof: First consider the case where $N'$ is indecomposable. Assume $f'$ is not an isomorphism. Then $\mu(N) \cup \{|N'|, |M|\} \leq \mu(M)$. However, by assumption $\mu(M) = \mu(N) \cup \{|M|\}$ and $\mu(N) \cup \{|M|\} < \mu(N) \cup \{|N'|, |M|\}$, a contradiction. Now, consider the general case: Write $N' = \bigoplus_i N_i$ with indecomposable modules $N_i$. The Main Property (a) asserts that $\mu(N) \leq \max \mu(N_i)$ and trivially $\max \mu(N_i) \leq \mu(M)$. Since $\mu(M)$ starts with $\mu(N)$, the same is true for $\max \mu(N_i)$, thus by (b$'$) there is some $j$ such that the map $\pi_j f'$ is injective, where $\pi_j\colon N' \to N_j$ is the canonical projection. Besides the monomorphisms $\pi_j f'\colon N \to N_j$, there also exists a monomorphism $N_j \to N' \to M$. Since the latter is a proper monomorphism, and $N_j$ is indecomposable, we are in the first case, thus we know that $\pi_j f'$ is an isomorphism, thus $f'$ is a split monomorphism.

The property of the inclusion map $f\colon N \to M$ in Corollary 2 may be called *mono-irreducibility*, in parallelity to the Auslander-Reiten notion of irreducibility: $f$ is a non-invertible monomorphism and any factorization $f = f''f'$ of $f$ using monomorphisms $f'$ and $f''$ implies that $f'$ is a split monomorphism of $f''$ is a split epimorphism (thus isomorphism). Irreducible monomorphism are mono-irreducible; however there are obvious mono-irreducible maps which are not irreducible: for example consider the path algebra of the quiver

$$\begin{tikzcd}[row sep=small] a & \\ & c \arrow[ul] \arrow[dl] \\ b & \end{tikzcd}$$

The inclusion map of the simple module $S(a)$ into its injective envelope is mono-irreducible, however it factors through the projective cover of $S(c)$, thus it is not irreducible. Also, there is the following phenomenon: Given indecomposable modules

$X, Y$, there may be irreducible monomorphisms $f\colon X \to Y$ and also a monomorphism $g\colon X \to Y$ which is not even mono-irreducible. For example, take the hereditary algebra $\widetilde{A}_{21}$, let $S$ be simple projective and $P$ the indecomposable projective of length 4. Then $\mathrm{Hom}(S, P)$ is 2-dimensional and the non-zero maps are monomorphisms. Thus the monomorphisms (up tp scalar multiplication) $S \to P$ form a projective line; one of these equivalence classes is not mono-irreducible (it factors through an indecomposable length 2 submodule), the remaining ones are irreducible, thus mono-irreducible.

**Corollary 3.** *Let $N$ be a Gabriel-Roiter submodule of the indecomposable module $M$. Then $M/N$ is indecomposable.*

Proof of Corollary 3: Assume $M/N = Q_1 \oplus Q_2$ with non-zero modules $Q_1, Q_2$. For $i = 1, 2$, write $Q_i = N_i/N$, where $N \subset N_i \subset M$. According to Corollary 2, we find submodules $N_i'$ of $N_i$ such that $N_i = N \oplus N_i'$. Then $M = N \oplus N_1 \oplus N_2$, in contrast to the fact that $M$ is indecomposable.

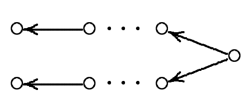
This corollary asserts, in particular, that *any indecomposable module $M$ of length at least 2 occurs as the middle term of an exact sequence*

$$0 \to N \to M \to M/N \to 0,$$

*where all three terms $N, M, M/N$ are indecomposable.* (This exact sequence has the following additional property: its equivalence class in $\mathrm{Ext}^1(M/N, N)$ is annihilated by the radical of $\mathrm{End}(M/N)$, where we view $\mathrm{Ext}^1(M/N, N)$ as usual as a right $\mathrm{End}(M/N)$-module.)

Also we see: *If $M$ and $N$ are indecomposable modules with $|N| < |M|$ and $\mu(M) = \mu(N) \cup \{|M|\}$, then the cokernel of* **any** *monomorphism $f\colon N \to M$ is indecomposable.* One should be aware that there are plenty of pairs of modules $N, M$ such that there do exist monomorphisms $f_1, f_2\colon N \to M$ such that the kernel of $f_1$ is indecomposable whereas the kernel of $f_2$ is not (for example, let $\Lambda$ be the path algebra of the Kronecker quiver and let $N, M$ be preprojective $\Lambda$-modules of length 1 and 5, respectively).

One may wonder about the possible modules which occur as factor modules $M/N$, where $M$ is indecomposable and $N$ is a Gabriel-Roiter submodule. For the path algebra of a quiver of type $A_n$, all these factors are serial and of length at most $\frac{n+1}{2}$, a factor of length $\frac{n+1}{2}$ occurs for the sincere representation of the quiver of type $A_n$ ($n$ odd) with a unique source



and with arms of equal length.

## 4. Main Results.

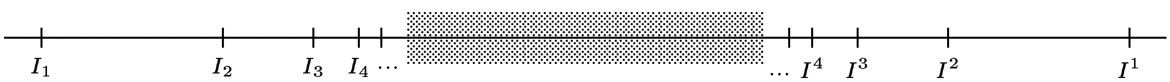
The indecomposable $\Lambda$-modules of length at most $n$ belong to the classes $\mathcal{A}(I)$ with $I \subseteq \{1, 2, \ldots, n\}$, and there are just finitely many such classes. Thus as soon as we exhibit (as we will do now) an infinite list of Gabriel-Roiter measures for $\Lambda$, this implies that $\Lambda$ cannot be of bounded representation type. Thus, the following theorem strengthens the assertion of the first Brauer-Thrall conjecture. In contrast to the assertion of the first Brauer-Thrall conjecture, the statement is meaningfull even in case $\Lambda$ is a finite ring (i.e. a ring with finitely many elements). Recall that a Gabriel-Roiter measure $I$ is said to be *of finite type* provided there are only finitely many isomorphism classes in $\mathcal{A}(I)$.

**Theorem 1.** *Let $\Lambda$ be of infinite representation type. Then there are Gabriel-Roiter measures $I_t, I^t$ for $\Lambda$ with*

$$I_1 < I_2 < I_3 < \quad \cdots \quad < I^3 < I^2 < I^1$$

*such that any other Gabriel-Roiter measure $I$ for $\Lambda$ satisfies $I_t < I < I^t$ for all $t \in \mathbb{N}_1$, and all these Gabriel-Roiter measures $I_t$ and $I^t$ are of finite type.*

We call the modules in $\bigcup_t \mathcal{A}(I_t)$ (or the additive category with these indecomposable modules) the *take-off part* of the category $\mathcal{A}$, and $\bigcup_t \mathcal{A}(I^t)$ (or the additive category with these indecomposable modules) the *landing part* of $\mathcal{A}$. The remaining indecomposables (those which do not belong to the take-off part or the landing part) are said to form the *central part.* It is the central part which should be of particular interest in future:




Note that *for any $n$, there are only finitely many isomorphism classes of indecomposable modules of length $n$ which belong to the take-off part* (since they belong to only finitely many classes $\mathcal{A}(I_t)$ and any class $\mathcal{A}(I_t)$ is of finite type). Similarly, *there are only finitely many isomorphism classes of indecomposable modules of length $n$ which belong to the landing part.*

It is obvious that the modules in $\mathcal{A}(I_1)$ are just the simple modules, those in $\mathcal{A}(I_2)$ are the local modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(I^1)$ are the indecomposable injective modules of largest possible length. For general $t$, it seems to be difficult to characterize the modules in $\mathcal{A}(I_t)$ or $\mathcal{A}(I^t)$ in a direct way.

Recall that Auslander-Smalø have introduced in [AS] the notion of preprojective and preinjective modules (actually with reference to the work of Roiter and Gabriel).

**Theorem 2.** *The modules in the landing part are preinjective.*

Since modules which have infinitely many different Gabriel-Roiter measures cannot have bounded length, we obtain in this way a new proof for the assertion that the indecomposable preinjective modules are of unbounded length ([AS],5.11). But note that usually there will exist preinjective indecomposables which do not belong to the landing part. For example, any simple module belongs to $\mathcal{A}(I_1)$, thus a simple injective module is preinjective and in the take-off part, thus not in the landing part. Also, there may exist preinjective modules $Q$ such that $\mathcal{A}(\mu(Q))$ is infinite, as the example of the radical-square-zero algebra with quiver

$$\circ \longleftarrow \circ \leftleftarrows \circ$$

shows: take for $Q$ the indecomposable injective module of length 2. But there may be even infinitely many isomorphism classes of preinjective indecomposables which do not belong to the landing part:

**Example.** Consider the tame hereditary algebra of tye $\widetilde{A}_{21}$

$$\begin{tikzcd} & b \arrow[dl] & \\ a & & c \arrow[ul] \arrow[ll] \end{tikzcd}$$

For a tame hereditary algebra, the Auslander-Smalø preinjectives are just those modules which belong to the preinjective component.

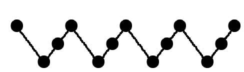
We denote by $S(x)$ the simple module corresponding to the vertex $x$, thus $S(a)$ is projective and $S(c)$ is injective. The top composition factors of the preinjective indecomposable modules are injective, all but at most one socle composition factors are projective, the exceptional one will be of the form $S(b)$. Now, in case the socle is projective, then the GR-measures are as follows:

$$\cdots > 1235689, 10 > 123567 > 1234,$$

the general form is

$$123|56|89|\cdots|3i-1, 3i|\cdots|3n-1, 3n|3n+1,$$

with $n \geq 0$. For $n = 4$, it looks as follows



and for $n \geq 1$, the GR-filtration starts with $M_1 \subset M_2 \subset M_3$, where $M_3$ is the indecomposable length 3 module seen left: it is uniform, but not serial.

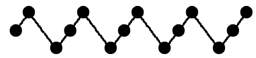
On the other hand, those preinjectives with $S(b)$ in the socle have GR-measure

$$123|6|9|\cdots|3i|\cdots|3n|3n+2,$$

with $n \geq 0$. For small $n \geq 1$, we obtain the values

$$\cdots > 12369, 11 > 12368 > 1235.$$

Here is the picture for $n = 4$



now, for $n \geq 1$, the GR-filtration starts with $M_1 \subset M_2 \subset M_3$, where $M_3$ is the serial length 3 module seen right.

It follows that all the preinjective modules with $S(b)$ in the socle belong to the central part.

In contrast to Theorem 3, the modules in the take-off part are usually **not** preprojective. Here is an example: Let $\Lambda = k[X,Y]/\langle XY, X^3, Y^3\rangle$ and $A$ the ideal generated by $X^2$ and $Y^2$ (these elements actually form a basis of $A$). The take-off part for $\Lambda$ is the same as the take-off part for $\Lambda/A$ and these modules are the preprojective $\Lambda/A$-modules, but none of them is preprojective as a $\Lambda$-module.

Note that there is no dualization principle concerning the take-off and the landing part (whereas the notions of preprojectivity and the preinjectivity are dual ones)! If we want to invoke dual considerations, then we have to work with a corresponding Gabriel-Roiter comeasure which is based on looking at indecomposable **factor** modules in contrast to the Gabriel-Roiter measure which is based on indecomposable **sub**modules. This will be done in section 7.

It is usually difficult to specify the position of the possible Gabriel-Roiter measures. But here is such an assertions, dealing with uniform modules:

**Proposition.** *Let $I^1 = (1, 2, ..., t)$ and $1 \leq s < t$. Assume the following: for any simple $\Lambda$-module with injective envelope $Q(S)$ of length greater than s, there are only finitely many indecomposable $\Lambda$-modules with a submodule of the form $S$. Then $(1, 2, \dots, s)$ is a landing measure.*

Proof: We show that any indecomposable module $M$ with $\mu(M) > (1, 2, \dots, s)$ has a composition factor of the form $S$, such that $|Q(s)| > s$. Thus assume that $\mu(M) > (1, 2, \dots, s)$ and take a Gabriel-Roiter-filtration of $M$. The first $s$ submodules in the filtration are uniform of length $i$ with $1 \leq i \leq s$. In particular, $M$ contains a uniform module $U$ of length $s$. Let $S$ be its socle, thus $U$ embeds into $Q(S)$, and this is a proper embedding, since otherwise $U = Q(S)$ would be a direct summand of $M$. However $M$ is indecomposable and of length greater than $s$. This shows that $|Q(S)| > s$ and $S \subseteq U \subseteq M$ is a submodule of $M$. By assumption, there are only finitely many such isomorphism classes. This shows that there are only finitely many isomorphism classes of indecomposable modules $M$ with $\mu(M) > (1, 2, \dots, s)$, thus $(1, 2, \dots, s)$ belongs to the landing part.

## 5. Infinitely generated modules.

Up to now, we have concentrated on $\Lambda$-modules of finite length, however the Gabriel-Roiter measure was introduced above for all $\Lambda$-modules $M$, not just those of finite length. Note that by definition $\mu(M)$ is the supremum of $\mu(M')$, where $M'$ are the finitely generated submodules of $M$ (or just the indecomposable ones).

We extend the notion of a Gabriel-Roiter filtration as follows: In case there exists a (countable) chain of submodules

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup\nolimits_i M_i = M \quad \text{such that} \quad \mu(M) = \{|M_i| \mid i\},$$

then we call this chain a *Gabriel-Roiter filtration* of $M$. Of course, a finitely generated $\Lambda$-module $M$ has a Gabriel-Roiter filtration if and only if $M$ is indecomposable. As a consequence of Gabriel's Main Property we show now that also any infinitely generated module with a Gabriel-Roiter filtration is indecomposable:

**Corollary 4.** *Any module $M$ with a Gabriel-Roiter filtration is indecomposable.*

Proof: We can assume that there is given an infinite chain

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup\nolimits_i M_i = M$$

such that $M_i$ is a Gabriel-Roiter submodule of $M_{i+1}$, for all $i \geq 1$. Assume that there is given a direct decomposition $M = U \oplus V$ with $U, V$ both nonzero. Note that if $U \cap M_i = 0$ for all $i$, then $U = U \cap M = U \cap (\bigcup M_i) = \bigcup (U \cap M_i) = 0$. This shows that there is some index $s$ such that $U \cap M_s \neq 0$ and also $V \cap M_s \neq 0$. Choose finitely generated submodules $U' \subseteq U$ and $V' \subseteq V$ such that $M_s \subseteq M' = U' \oplus V'$, and decompose $U' = \bigoplus U_i$, $V' = \bigoplus V_j$ with indecomposable modules $U_i$ and $V_j$. Finally, choose $t$ such that $M' \subseteq M_t$.

Now we consider the Gabriel-Roiter measures: We get

$$\mu(M_s) \leq \max\{\mu(U_i), \mu(V_j)\} \leq \mu(M_t)$$

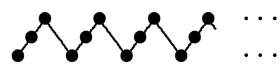
(the first inequality is Main Property (a), the second is trivial). Since $M_s$ and $M_t$ belong to a Gabriel-Roiter filtration, it follows that $\mu(M_t)$ starts with $\mu(M_s)$, thus also $\max\{\mu(U_i), \mu(V_j)\}$ starts with $\mu(M_s)$ and we can apply Main Property (b). Without loss of generality, we can assume that the composition of the inclusion $M_s \to \bigoplus_i U_i \oplus \bigoplus_j V_j = M'$ and the projection $\pi_1^U \colon M' \to U_1$ is injective (where $i = 1$ is one of the indices). Recall that there is a non-zero element $v \in V \cap M_s$. Since $M_s \subseteq M' = U' \oplus V'$, we can write $v = u' + v'$ with $u' \in U'$ and $v' \in V'$. However $u' = v - v' \in U' \cap V = 0$ shows that $v = v'$ belongs to $V'$. Since $v$ belongs to $V' = \bigoplus V_j$, it is mapped under $\pi_1^U$ to zero. This contradicts the fact that $\pi_1^U$ is injective.

**Theorem 3.** *Let $\Lambda$ be of infinite representation type. There do exist modules which have an infinite Gabriel-Roiter filtration*

$$M_1 \subset M_2 \subset \cdots \subseteq \bigcup\nolimits_i M_i = M$$

*such that all the modules $M_i$ belong to the take-off part.*

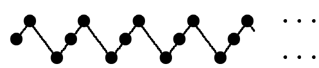
Note that according to Corollary 4, such a module $M$ is indecomposable. Also, any finitely generated submodule $M'$ of $M$ is contained in some $M_t$, thus belongs to the take-off part. In particular, for any natural number $n$, $M$ has only finitely many isomorphism classes of submodules of length $n$. In general, Theorem 3 will provide a large number of indecomposable $\Lambda$-modules $M$, however all these modules have the same Gabriel-Roiter measure! For example, if $K$ is the Kronecker quiver and $k$ is a countable and algebraically closed field, then all the "torsionfree $kK$-modules of rank 1" (see [Ri2]) occur in this way, and $\mu(M) = \{1, 2, 4, 6, 8, \dots\}$. On the other hand, for the tame algebra of type $\widetilde{A}_{21}$, there is only one such module $M$, namey the string module corresponding to



its Gabriel-Roiter measure is $\{1, 2, 4, 5, 7, 8, \dots\}$.

The existence of infinitely generated indecomposables for any artin algebra of infinite representation type was first shown by Auslander [A]. For a discussion of the question whether a union of a chain of indecomposable modules of finite length is indecomposable or not, we refer to [Ri1].

Let us note that there are indecomposable modules without a Gabriel-Roiter filtration. Of course, any module with a Gabriel-Roiter filtration is countably generated, here is an example of a countable generated indecomposable module without a Gabriel-Roiter filtration: We consider again the tame hereditary algebra of type $\widetilde{A}_{21}$ and take the Prüfer module for the simple module $S(b)$ which is neither projective nor injective:



its Gabriel-Roiter measure is {1,2,4,5,7,8,...}, but there is no corresponding sequence of submodules which exhaust all of $M$.

We have introduced above an embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$. In order to deal also with modules with are not finitely generated, we consider the set $\mathcal{P}_l(\mathbb{N}_1)$ of all subsets $I$ of $\mathbb{N}_1$ such that for any $n \in \mathbb{N}_1$, there is $n' \geq n$ with $n' \notin I$.

**Lemma 2.** *The Gabriel-Roiter measure $\mu(M)$ of any module $M$ belongs to $\mathcal{P}_l(\mathbb{N}_1)$.*

Proof. There is $m \in \mathbb{N}_1$ such that any indecomposable injective $\Lambda$-module has length at most $m$. Let $\mu(M) = \{a_1 < a_2 < \cdots < a_i < \cdots\}$ and assume that for some $n$ we have $a_{n+t} = a_n + t$ for all $t \in \mathbb{N}_1$. Let $s = m \cdot a_n$

There is a chain of indecomposable submodules $M_1 \subset M_2 \subset \cdots \subset M_{n+s}$ with $|M_i| = a_i$ for $1 \leq i \leq n+s$. Since $|M_{n+t}| = a_{n+t} = a_{n+t-1} + 1 = |M_{n+t-1}| + 1$, we see that $M_{n+t-1}$ is a maximal submodule of $M_{n+t}$. Since $M_{n+t}$ is indecomposable, the socle of $M_{n+t}$ has to be contained in $M_{n+t-1}$. Inductively, we see that the socle of $M_{n+t}$ is contained in $M_n$, for any $t \geq 1$, in particular, the socle of $M_{n+s}$ is contained in $M_n$, thus $M_{n+s}$ can be embedded into the injective envelope of $M_n$. Since any indecomposable injective module is of length at most $m$, the injective envelope of $M_n$ has length at most $m \cdot a_n$, thus $|M_{n+s}| \le m \cdot a_n$. But $|M_{n+s}| = |M_n| + s = (m+1)a_n > m \cdot a_n$, a contradiction.

The embedding of $\mathcal{P}_f(\mathbb{N}_1)$ into $\mathbb{Q}$ (thus into $\mathbb{R}$) extends to an embedding of $\mathcal{P}_l(\mathbb{N}_1)$ into the real interval $[0,1]$:

**Lemma 1′.** *The map $r\colon \mathcal{P}_l(\mathbb{N}_1) \to \mathbb{R}$ given by $r(I) = \sum_{i\in I} \frac{1}{2^i}$ for $I \in \mathcal{P}_l(\mathbb{N}_1)$ is injective, its image is contained in the interval $[0,1]$ and it preserves and reflects the ordering.*

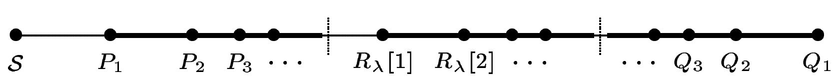
Remark: The map $r$ can be defined not just on $\mathcal{P}_l(\mathbb{N}_1)$, but on all of $\mathcal{P}(\mathbb{N}_1)$, however it will no longer be injective (indeed, for any element $I$ in $\mathcal{P}(\mathbb{N}_1) \setminus \mathcal{P}_l(\mathbb{N}_1)$, there is a unique finite set $I'$ with $r(I) = r(I')$). Of course, one easily may change the definition of $r$ in order to be able to embed all of $\mathcal{P}(\mathbb{N}_1)$ into $\mathbb{R}$: just use say 3 instead of 2 in the denominator. However, our interest lies in the Gabriel-Roiter measures which occur for finite dimensional algebras and Lemma 2 assures us that the definition of $r$ as proposed is sufficient for these considerations.

## 5. Examples.

**Example 1. The Kronecker quiver $\widetilde{A}_{11}$.** We have referred to this quiver already several times, it has vertices $a, b$ and two arrows $b \to a$; its representations are called *Kronecker modules.* There are two simple Kronecker modules, the projective simple module $S(a)$ and the injective simple module $S(b)$. If $M$ is a Kronecker module, its *dimension vector* is of the from $\mathbf{dim}\, M = (d_a, d_b)$, where $d_a$ is the Jordan-Hölder multiplicity of $S(a)$, and $d_b$ that of $S(b)$. The dimension vectors of the indecomposable modules are of the form $(x, y)$ with $|x-y| \le 1$. Here is the complete list of the indecomposable representations in case $k$ is algebraically closed:

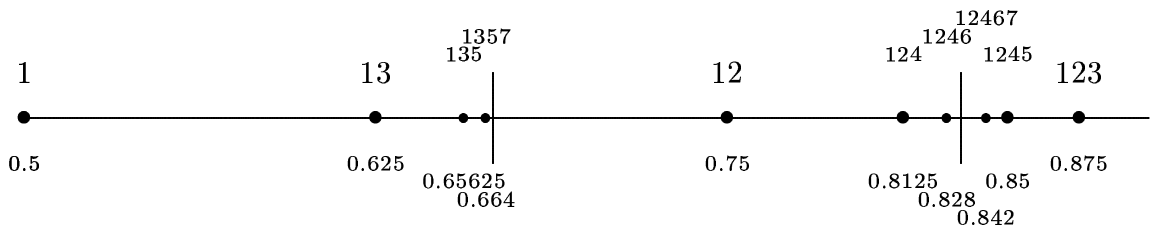
- The preprojectives $P_n$ for $n \in \mathbb{N}_0$, with $\mathbf{dim}\, P_n = (n+1, n)$ and $\mu(P_n) = \{1, 3, 5, \ldots, 2n+1\}$.
- The preinjectives $Q_n$ for $n \in \mathbb{N}_0$, with $\mathbf{dim}\, Q_n = (n, n+1)$ and $\mu(Q_n) = \{1, 2, 4, 6, \ldots, 2n, 2n+1\}$.
- The regular modules $R_\lambda[n]$ for $\lambda \in \mathbb{P}^1(k)$ and $n \in \mathbb{N}_1$, with $\mathbf{dim}\, R_\lambda[n] = (n, n)$ and $\mu(R_\lambda[n]) = \{1, 2, 4, 6, \ldots, 2n\}$.

The totally ordered set of all the Gabriel-Roiter measures for the Kronecker quiver looks as follows:




Here $\mathcal{S} = \mathcal{A}(\{1\}) = \{S(a), S(b)\}$. Note that there are precisely two accumulation points, indicated by the dotted vertical lines, they correspond to the only two Gabriel-Roiter measures for infinitely generated modules: to the left, there is $\{1, 3, 5, 7, \ldots\}$, this is the Gabriel-Roiter measure for all indecomposable torsionfree modules; to the right, there is $\{1, 2, 4, 6, 8, \ldots\}$, this is the Gabriel-Roiter measure for the so-called Prüfer modules (an account of the structure theory for infinitely generated Kronecker modules can be found for example in [Ri2]).

A more precise picture of the Gabriel-Roiter measures for the Kronecker algebra is the following; here the upper sequences are the measures $I$, the lower numbers the corresponding values $r(I)$:




In case $k$ is not algebraically closed, we have to take into account field extensions of $k$, or better indecomposable $k[T]$-module of finite length $N$, where $k[T]$ is the polynomial ring over $k$ in one variable $T$. Any indecomposable $k[T]$-module $N$ of length $n$ and with a simple submodule of dimension $d$ gives rise to a regular Kronecker module with dimension vector $(nd, nd)$ and Gabriel-Roiter measure $\{1, 3, 5, \dots, 2d{-}1, 2d; 4d, 6d, \dots, 2nd\}$. Thus we see that *the Gabriel-Roiter measure for the path algebra $k\Delta$ of a quiver $\Delta$ may depend on $k$* (and usually will).

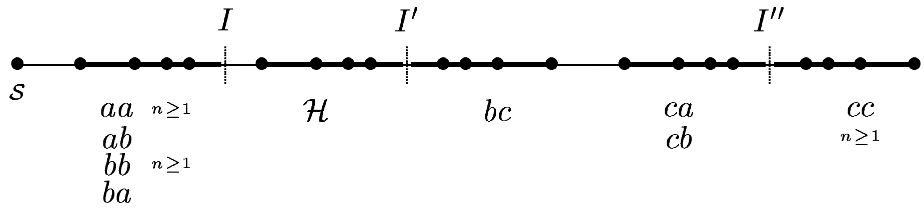
**Example 2. The tame hereditary algebra of type $\widetilde{A}_{21}$.** Also this algebra has been referred to before, we want to stress here some features which one should be aware of. In order to list all the indecomposable $\Lambda$-modules, we use that $\Lambda$ is a string algebra. Thus the indecomposable modules are the string and the band modules. Again, we restrict to the case of $k$ being algebraically closed.

There is a unique one-parameter family of band modules; they are of the form $R_\lambda[n]$, where $\lambda \in k \setminus \{0\}$ and $n \in \mathbb{N}_1$, with Gabriel-Roiter measure $\mu(R_\lambda) = \{1, 2, 3; 6, 9, \cdots, 3n\}$.

In order to write down the string modules, we use words in $\alpha, \beta, \gamma^{-1}$; the relevant distinction is given by fixing the vertices $x, y$ such that the word starts in $x$ and ends in $y$ (always $n \in \mathbb{N}_0$):

| $xy$ | **property** | **dimension** | **GR-measure** |
|---|---|---|---|
| $aa$ | preprojective | $3n{+}1$ | $1, 2, 4, 5, 7, 8, ..., 3n{-}2, 3n{-}1, 3n{+}1$ |
| $ab$ | preprojective | $3n{+}2$ | $1, 2, 4, 5, 7, 8, ..., 3n{-}2, 3n{-}1, 3n{+}1, 3n{+}2$ |
| $ac$ | homogeneous | $3n{+}3$ | $1, 2, 3; 6, 9, ..., 3n$ |
| $ba$ | regular, non-homog. | $3n{+}3$ | $1, 2, 4, 5, 7, 8, ..., 3n{-}2, 3n{-}1, 3n{+}1, 3n{+}3$ |
| $bb$ | regular, non-homog. | $3n{+}1$ | $1, 2, 4, 5, 7, 8, ..., 3n{-}2, 3n{-}1, 3n{+}1$ |
| $bc$ | preinjective | $3n{+}2$ | $1, 2, 3; 6, 9, ..., 3n; 3n{+}2$ |
| $ca$ | regular, non-homog. | $3n{+}2$ | $1, 2, 3; 5, 6, 8, 9, ..., 3n{-}1, 3n, 3n{+}2$ |
| $cb$ | regular, non-homog. | $3n{+}3$ | $1, 2, 3; 5, 6, 8, 9, ..., 3n{+}2, 3n{+}3$ |
| $cc$ | preinjective | $3n{+}1$ | $1, 2, 3; 5, 6, 8, 9, ..., 3n{-}1, 3n; 3n{+}1$ |

The set of Gabriel-Roiter measures for $\Lambda$ has the following structure:




Here, $\mathcal{H}$ denotes the class of all homogeneous modules (the bands as well as the strings of type $ac$), whereas $\mathcal{S}$ are the simple modules.

Some observations:

(1) There are many **"maximal"** GR-measures $I$ (maximality should mean that no other GR-measure starts with $I$), in particular see $ba$, but also $bc$ and $cc$.

(2) The **take-off part** contains all the preprojective modules, but in addition also half of the non-homogeneous tube (namely all the regular modules which have the simple module $S(b)$ as submodule.

(3) The **landing part** contains only half of the preinjective modules (also the modules $bc$ are preinjective)

(4) The GR-measure apparently does not distinguish modules which have quite different behaviour, see $aa$ and $bb$ (however, $aa$ and $bb$ will be distinguished in case we invoke the dual concepts, see the next appendix)

(5) There are three **accumulation points** $I, I', I''$:

$$I = \{1, 2, 4, 5, 7, 8, 10, 11, \dots\}$$
$$I' = \{1, 2, 3, 6, 9, 12, 15, \dots\}$$
$$I'' = \{1, 2, 3, 5, 6, 8, 9, 11, 12, \dots\}$$

The first one $I$ is the Gabriel-Roiter measure of the torsionfree modules; $I'$ is the Gabriel-Roiter measure for all the Prüfer modules arising from homogeneous tubes; $I''$ is that of the Prüfer module containing the 2-dimensional indecomposable regular module as a submodule.

(6) There is one additional Prüfer module, it contains the simple module $S(b)$ as a submodule: this module does **not** have a Gabriel-Roiter filtration!

### 6. Dualization

**Dualization.** Almost all the considerations presented above can be dualized and then they yield corresponding dual results. This means that instead of looking at filtrations

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

with $M_i$ indecomposable for $1 \le i \le t$, we now look at such filtrations with $M/M_{i-1}$ indecomposable for $1 \le i \le t$. We prefer to use now the opposite order on $\mathcal{P}(\mathbb{N}_1)$, we denote it by $\le^*$ (and $<^*$), thus $I \le^* J$ iff $J \le I$. For a (not necessarily finitely generated) $\Lambda$-module $M$, let $\mu^*(M)$ be the infimum of the sets $\{|M_1|, \dots, |M_t|\}$ in $(\mathcal{P}(\mathbb{N}_1), \leq^*)$, where $M_1 \subset M_2 \subset \cdots \subset M_t$ is a chain of submodules of $M$ with $M/M_{i-1}$ indecomposable for $1 \leq i \leq t$, we call $\mu^*(M)$ the *Gabriel-Roiter comeasure* of $M$. We say that $J$ is a Gabriel-Roiter comeasure for $\Lambda$ provided there exists an indecomposable module $M$ with $\mu^*(M) = J$.

In order to visualize $(\mathcal{P}_l(\mathbb{N}_1), \leq^*)$, we use the embedding $r^*\colon (\mathcal{P}_l(\mathbb{N}_1), \leq^*) \to \mathbb{R}$ given by $r^*(I) = -r(I)$. Note that for any non-zero module $M$, we have $-1 \leq r^*(\mu(M)) \leq 0$. (Actually, it may be advisable to rescale $r$ and $r^*$ so that $r(\{1\}) = r^*(\{1\}) = 0$ and $r(\mathbb{N}_1) = 1, r^*(\mathbb{N}_1) = -1$.)

The dual version of Main Property reads as follows:

**Main Property*.** *Let $Y_1, \dots, Y_t, Z$ be indecomposable $\Lambda$-modules of finite length and assume that there is an epimorphism $g\colon \bigoplus_{i=1}^t Y_i \to Z$.*

(a) *Then $\max \mu^*(Y_i) \leq^* \mu^*(Z)$.*
(b) *If $\max \mu^*(Y_i)$ starts with $\mu^*(Z)$, then there is some $j$ such that $gu_j$ is surjective, where $u_j\colon Y_j \to \bigoplus_i Y_i$ is the canonical inclusion.*
(b′) *If $\mu^*(Z) = \max \mu^*(Y_i)$, then $g$ splits.*

As a consequence, we see that the class of modules which are direct sums of modules $M$ with $I \leq^* \mu^*(M)$ for some set $I \subseteq \mathbb{N}_1$ is closed under factor modules. In this way, one obtains a second interesting filtration of the category of all $\Lambda$-modules by subcategories, now these subcategories are closed under factor modules.

Let us formulate the dual versions of Theorem 1 and Theorem 2:

**Theorem 1*.** *Let $\Lambda$ be of infinite representation type. Then there are Gabriel-Roiter comeasures $J_t, J^t$ for $\Lambda$ with*

$$J_1 < J_2 < J_3 < \quad \cdots \quad < J^3 < J^2 < J^1$$

*such that any other Gabriel-Roiter comeasure $J$ for $\Lambda$ satisfies $J_t < J < J^t$ for all $t \in \mathbb{N}_1$, and all these Gabriel-Roiter comeasures $J_t$ and $J^t$ are of finite type.*

We do not have a suggestion how to call the modules in $\bigcup_t \mathcal{A}(J_t)$ or in $\bigcup_t \mathcal{A}(J^t)$. The indecomposable modules which belong neither to $\bigcup_t \mathcal{A}(J_t)$ nor to $\bigcup_t \mathcal{A}(J^t)$ may be said to be form the $*$-*central part.*

Note that for any $n$, there are only finitely many isomorphism classes of indecomposable modules of length $n$ which belong to $\bigcup_t \mathcal{A}(J_t)$ or to $\bigcup_t \mathcal{A}(J^t)$.

The modules in $\mathcal{A}(J^1)$ are just the simple modules, those in $\mathcal{A}(J^2)$ are the uniform modules of Loewy length 2 of largest possible length. On the other hand, the modules in $\mathcal{A}(J_1)$ are the indecomposable projective modules of largest possible length.

**Theorem 2*.** *The modules in $\bigcup_t \mathcal{A}(J_t)$ are preprojective.*

There does not seem to exist a dual version of Theorem 3, since Theorem 3 deals with infinitely generated modules. It is the assertion of Corollary 4 which breaks down. For example, consider again the Kronecker quiver and let $Q_n$ be the preinjective module of length $2n+1$. Then $Q_{n-1}$ is a Gabriel-Roiter factor module of $Q_n$, for $n \geq 1$, and the sequences of epimorphisms

$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0$$

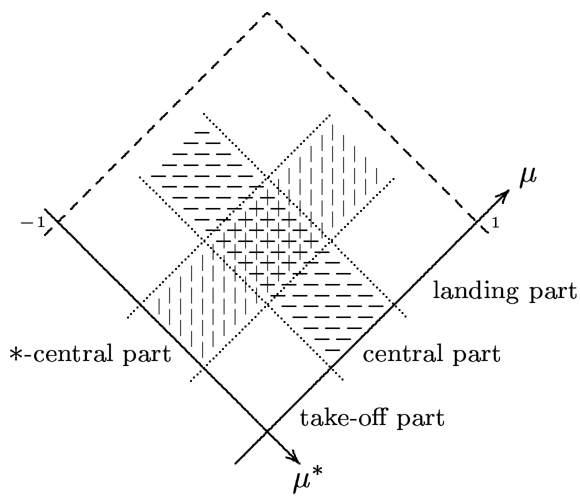
may be called Gabriel-Roiter cofiltrations. If we form the inverse limits, we obtain infinite direct sums of Prüfer modules; in particular, such an inverse limit module is not indecomposable.

## 7. The Rhombic Picture.

We are going to use now both the measure and the comeasure at the same time. Given a pair $(J, I)$ of finite subsets $I, J$ of $\mathbb{N}_1$, we may consider the module class

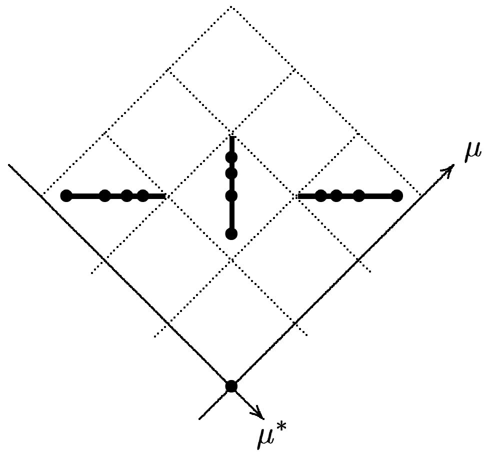
$$\mathcal{A}(J, I) = \{M \mid M \quad \text{indecomposable}, \quad \mu^*(M) = J,\ \mu(M) = I\},$$

thus we attach to a module $M$ the pair $(\mu^*(M), \mu(M))$. The possible pairs $(J, I)$ can be considered (via $r^*$ and $r$) as elements in the rational plane $\mathbb{Q}^2$ :



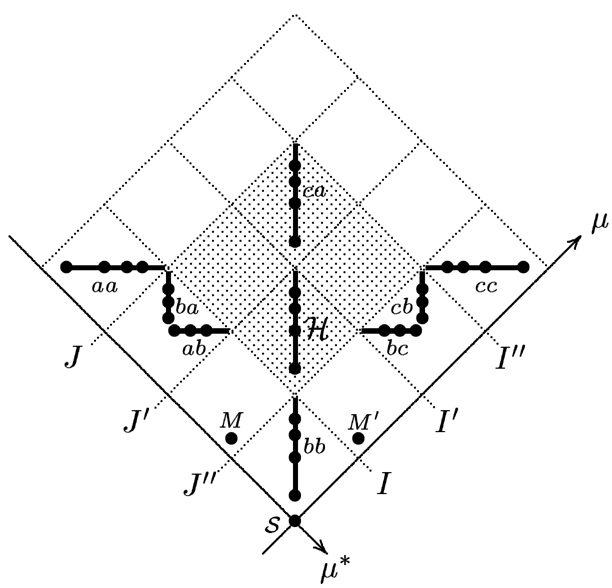

The horizontally dashed region is the central part (in between the take-off part and the landing part); the vertically dashed region is the $*$-central part. The main information one should keep in mind: *The only possible pairs $(J, I)$ of finite subsets of $\mathbb{N}_1$ such that $\mathcal{A}(J, I)$ contains infinitely many isomorphism classes, are those which belong both to the central and the $*$-central part.*

**Example 1: The Kronecker quiver,** with $k$ algebraically closed. The picture which we obtain is nearly the same as the commonly accepted visualization, the only exception being the position of the simple modules. One should be aware that the commonly accepted visualization with the preprojectives and the preinjectives being drawn horizontally and the tubes being drawn vertically in the middle was based mainly on the feeling that this arrangement reflects much of the structure of the category, but for the actual position of the individual modules there was no further mathematical justification. The rhombic picture should be seen as a definite reassurance in this case (but it suggests deviations in other cases).




Even for the Kronecker quiver, one should be aware that there does exist a deviation, namely the position of the simple modules. Of course, they are usually drawn far apart, one at the left end, the other at the right end, now they are located at the same position: in the middle lower corner. But note that the rhombic picture for the Kronecker quiver and the algebra $k[X,Y]/(X,Y)^2$ do not differ, and the usual Auslander-Reiten picture for the latter algebra puts its unique simple module precisely at this position (and bends down the preprojective modules on the left as well as the preinjective modules on the right to form half circles).

**Example 2. The tame hereditary algebra of type $\widetilde{A}_{21}$.** Here is the rhombic picture, for $k$ algebraically closed:

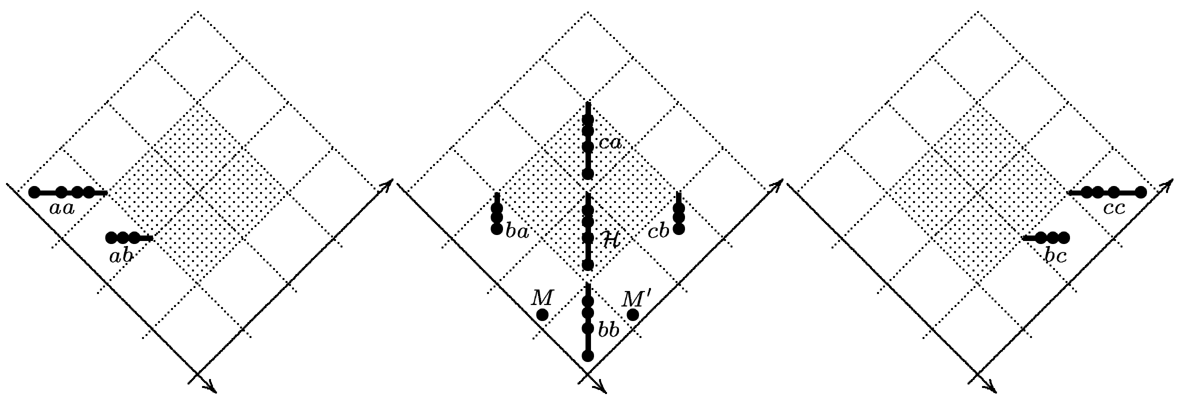



Two modules have to be specified separately, the indecomposable modules $M, M'$ of length 3 and Loewy length 2: $M$ is local, $M'$ uniform; note that $M$ has type $ba$,

$M'$ type $cb$. The accumulation points $I, I', I''$ for the Gabriel-Roiter measure are marked on the $\mu$-axis; similarly, the accumulation points $J, J', J''$ for the Gabriel-Roiter comeasure are marked on the $\mu^*$-axis (note that $J = I'', J' = I', J'' = I$ in $\mathcal{P}(\mathbb{N}_1)$). The intersection of the central and the $*$-central part has been dotted, this region contains for every $n \in \mathbb{N}_1$ a $\mathbb{P}^1(k)$-family of indecomposable representations of length $3n$.

One immediately realizes that the rhombic picture again corresponds quite well to the commonly used visualization, at least after deleting the simple modules. The preprojectives and the preinjectives are arranged horizontally, the regular modules vertically (there is one exceptional tube of rank 2, it has four types of indecomposable modules, namely the types $ca$, $ba$ (including $M$), $bb$ and $cd$ (inclusing $M'$). Let us take apart these three parts of the category:

Preprojectives Regular modules Preinjectives

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