ON RINGS WITH THE SAME SET OF PROPER IDEALS

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Abstract: We investigate pairs of rings with a set of common ideals.

In 1980's, a series of papers appeared in Canadian Journal of Mathematics ([1],[2]) that investigated pairs of commutative rings with the same set of prime ideals. We consider some generalizations of the study in the noncommutative setting. Throughout, all rings are assumed to be associative (but not necessarily commutative) with an identity element. The term "*subring*" will be used for a *unital subring*. Thus, not only a subring inherits its binary operations from its overring, but also they have the same identity element.

Consider $H = Hom_{\mathbb{R}}(V, V)$, where V is a vector space over $\mathbb{R}$ with $\dim_{\mathbb{R}}(V) = \aleph_{\omega_0}$ (ω_0 is the first limit ordinal). The center of H is isomorphic to $\mathbb{R}$ and hence, it has subfields K and F such that $K \not\subset F$ and $F \not\subset K$. Let $M = \{f \in Hom_{\mathbb{R}}(V, V) \mid \dim f(V) < \aleph_{\omega_0}\}$. Then $S = M + K$ and $R = M + F$ are an example of a pair of rings with the same set of prime ideals. Further more, S and R have infinitely many ideals and all of their proper ideals are prime ideals. A curiosity therefore arises for a pair of rings with the same set of proper ideals. By our first theorem, the only possible pairs of subrings of a commutative ring with the same set of proper ideals are fields.

Theorem 1. *Two distinct subrings R and S of a ring are division ring if and only if they have the same set of proper right ideals.*

Proof. Since $R \neq S$, they cannot have two distinct maximal right ideals in common. Let M be the unique maximal ideal of R and S, and suppose that $0 \neq a \in M$. Then, since R and S have the same set of proper right ideals, we have $aR = aS$. Further, since $1 - m$ is invertible for any $m \in M$, we must have $aR = aS \neq aM$. Thus aS/aM is a one-dimensional vector space over the division ring S/M and $aS/aM = aR/aM$ is also a one-dimensional vector space over R/M. This is a contradiction since $R/M \neq S/M$. Thus, $M = 0$ and hence R and S are division rings. □

We now state two propositions on a pair of rings with an ideal in common.

Proposition 1. *Let R and S be subrings of a ring and suppose that they have a common ideal I. If P is a prime ideal of R, then* $\tilde{P} = \{a \in S \mid IaI \subseteq P\}$ *is either S or a prime ideal of S.*

[1] The detailed version of this paper has been submitted elsewhere.

Proposition 2 *Let R and S be subrings of a ring having a common ideal I. If P is a primitive ideal of R, then $\tilde{P}=\{a\in S\mid IaI\subseteq P\}$ is either S or a primitive ideal of S.*

Our second theorem yields that a pair of rings has the same set of prime ideals if and only if they have the same set of maximal ideals. We denote the set of prime ideals of a ring R by *Spec(R);* the set of maximal ideals of a ring R by *Max(R);* and the set of primitive ideals of a ring R by $Prim(R)$.

Theorem 2. *Let $R\neq S$ be subrings of an arbitrary ring. Then the following statements are equivalent:*

(a) $Max(S)\supseteq Max(R)$
(b) $Max(S)\subseteq Max(R)$
(c) $Spec(S)=Spec(R)$
(d) $Prim(S)=Prim(R)$

Proof. If $Max(S)\supseteq Max(R)$, then R has a unique maximal ideal M. Let N be another maximal ideal of S. Then since $S=M+N$, there exist $m\in M$ and $n\in N$ such that $1=m+n$. But then $n=1-m\in R\setminus M$ and hence $RnR=R$. Hence, $M^2=MRnRM\subseteq N$. Since N is a prime ideal of S, this is a contradiction. Therefore, $Max(S)=Max(R)=\{M\}$. This shows the equivalence of the statement (a) and (b).
Suppose now that $Max(S)=Max(R)=\{M\}$ and let $P\neq M$ be a prime ideal of R. Then, by Proposition 1, $\tilde{P}=\{a\in S\mid MaM\subset P\}$ is a prime ideal of S. Since M is the unique maximal ideal of S, we have $\tilde{P}\subseteq M$, and so $\tilde{P}$ is an ideal of R. Since $M\tilde{P}M\subset P$, we obtain $\tilde{P}\subset P$, and therefore $P=\tilde{P}$ is a prime ideal of S. Since a primitive ideal is prime, the equivalence of the statement (a), (b), and (d) can be shown similarly by using Proposition 2. □

For a ring T, let $\mathbb{S}(T)$ be the set of all subrings S of T with $Spec(S)=Spec(T)$. We note that if T *is a ring with unique maximal ideal M, then*
$\mathbb{S}(T)=\{p^{-1}(S)|S \text{ is a simple subring of } T/M\}$ *where* $p:T\to T/M$ *is the canonical epimorphism.*

A ring is called *fully idempotent* if every ideal of R is idempotent. A commutative fully idempotent ring is Von Neumann regular. However, the class of fully idempotent rings strictly contains the class of regular rings.

Proposition 3. *let R and S be fully idempotent subrings of a ring. Then R and S have the same set of proper ideals if and only if R and S have the same set of prime ideals.*

We are in a position to give a few examples.

Example 1. *An example of a pair of rings having the same set of maximal (therefore prime) ideals but the set of proper ideals are not identical.*

Let $R = \mathbb{Q}\left(\sqrt{3}\right) \oplus \mathbb{R}$ and $S = \mathbb{Q}\left(\sqrt{2}\right) \oplus \mathbb{R}$ be additive abelian groups with multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$. Then R and S have a unique maximal ideal $M = 0 \oplus \mathbb{R}$. Let $I = 0 \oplus \mathbb{Q}\left(\sqrt{2}\right)$. Then I is an ideal of S but not of R. □

Example 2. *An example of a pair of rings that have a nonzero ideal in common but the set of prime ideals are not identical.*

Let K be a field, and $K[x]$ and $K[y]$ be two polynomial rings over K. Consider the ring $S = K[x] \oplus K[y]$ and its subring $R = \{(a + xf(x), a) \in S \mid a \in K, f(x) \in F[x]\}$. Then R and S have common ideal $I = \{(xf(x), 0) \mid f(x) \in K[x]\}$. Clearly $P = \{(0,0)\}$ is a prime ideal of R, but it is not a prime ideal of S. □

Example 3. *An example of a pair of rings that are not fully idempotent but have the same set of prime ideals.*

Let $\bar{R}$ be the ring consisting of countable matrices over $\mathbb{R}$ of the form

$$\begin{pmatrix} A_m & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix}$$

where $a \in \mathbb{R}$ and A_m is an arbitrary $m \times m$ matrix over $\mathbb{R}$ and m is allowed to be any integer.

Let $\bar{S} = \bar{M} + F$ where F is a subfield of the center of $\bar{R}$ and $\bar{M} = \begin{pmatrix} A_m & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}$.

Let $S = \bar{S} \oplus \bar{M}$ and $R = \bar{R} \oplus \bar{M}$ be additive abelian groups with multiplication defined by $(a, b)(c, d) = (ac, ad + bc)$. Then, S and R have the unique common maximal ideal $M = \{(m_1, m_2) \mid m_1, m_2 \in \bar{M}\}$ and hence, they have the same set of prime ideals. However, the ideal $I = \{(0, m) \mid m \in \bar{M}\}$ is not idempotent. □

Next, we investigate properties that pass through a pair of rings with common ideals. By Theorem 2, if two subrings R and S of a ring have the common maximal ideal, then they have the same set of prime ideals. Thus, in particular, if S is prime, then so is R. For a ring R, let $B(R)$ denote its prime radical, and $J(R)$ denote its Jacobson radical. Using Propositions 1 and 2, one can prove Lemma 1 below and hence Proposition 4 holds.

Lemma 1. *Let R and S be subrings of a ring having a common ideal I.*

(a) If $B(R) \subset I$, *then* $IB(S)I \cap I \subset B(R)$.
(b) If $J(R) \subset I$, *then* $IB(S)I \cap I \subset J(R)$.

Proposition 4. *Let R and S be subrings of a ring having a common ideal I.*

(a) If R *is a semiprime ring and if* $r_S(I) = \ell_S(I) = 0$, *then* S *is a semiprime ring.*
(b) If R *is a semiprimitive ring and if* $r_S(I) = \ell_S(I) = 0$, *then* S *is a semiprimitive ring.*

Let $R \subseteq S$ be rings with a common ideal I, and let P be a prime ideal of R with $I \not\subset P$. Then "lying over" holds, i.e., there exists a prime ideal Q in S such that $Q \cap R = P$. (See for example Rowen [4]).

Proposition 5. *Let* $R \subseteq S$ *be rings with a common ideal I. If P is a prime ideal of S with* $I \not\subset P$, *then* $P \cap R$ *is a prime ideal of* R.

Using Propositions 2, one can prove Lemma 2 below and hence Proposition 6 holds.

Lemma 2. *Let* $R \subseteq S$ *be rings with a common ideal I. If* $B(S) \subset I$, *then* $B(R) \cap I \subset B(S)$.

Proposition 6. *Let* $R \subseteq S$ *be rings with a common ideal* I. *Then if* S *is a semiprime ring and if* I *is an essential ideal of* S, *then R is a semiprime ring.*

A ring all of whose (two sided) ideal is idempotent is called a *fully idempotent ring.* A fully idempotent ring is in particular, a semiprime ring.

Proposition 7. *Let R and S be subrings of a ring having the common maximal ideal M. Then if R is fully idempotent, then so is S and in this case they have the same set of proper ideals.*

Every right ideal of a von Neumann regular is idempotent. A ring all of whose right ideal is idempotent is called a *fully right idempotent ring* and has received some attention in the literature.

Proposition 8. *Let R and S be subrings of a ring having the common maximal ideal M. Then if R is fully right idempotent, then so is S and in this case they have the same set of proper ideals.*

The next natural question is whether or not the "regularity" passes through two rings having the common maximal ideals.

Example 4. Let W denote the n-th Weyl algebra over a field of characteristic zero. It is well known that W is a simple Noetherian domain, and hence W is an Ore domain. Let D denote the filed of fraction of W. Let R be the set of countable matrices over D of the form

$$\begin{pmatrix} A_m & & & \\ & a & & \\ & & a & \\ & & & \ddots \end{pmatrix}$$

where $a \in D$ and A_m is an arbitrary $m \times m$ matrix over D and m is allowed to be any integer. Let S be the same set of matrices except $a \in W$. Then R and S have the unique maximal ideal M that consists of countable matrices of the form

$$\begin{pmatrix} A_m & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots \end{pmatrix}.$$

While it is easy to see that R is a von Neumann regular ring, S is not von Neumann regular since $S/M \simeq W$: a simple Northerian but not an Artinian ring. □

Neither the descending nor the ascending chain condition passes through a pair of rings with the same set of ideals in general.

Proposition 9. *Let $R \subset S$ be rings with the common maximal ideal. If S satisfies a polynomial identity, then R is right Noetherian if and only if S is right Noetherian and S/M is finitely generated right R/M-module.*

For a right R-module M and an ideal I of a ring R, consider $P_I(M) = \{m \in M \mid mI = 0\}$. M is said to be *split* in P_I if $P_I(M)$ is a direct summand of M, and P_I is said to be *splitting* if every R-module M *splits* in P_I. For a non-zero ideal I of a prime fully right idempotent ring R, Theorem 1.3 of Hirano-Tsutusi [3] yields that P_I is splitting if and only if R/I is semisimple Artinian. Therefore, Example 4 shows that "P_I splitting property" does not in general pass through a pair of rings with the same set of proper ideals.

Proposition 10. *Let R and S be non-prime subrings of a ring having the common maximal ideal M. Then* P_I *is splitting for every ideal I of R if and only if* P_I *is splitting for every ideal I of S.*

REFERENCES

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