FROBENIUS FULL MATRIX ALGEBRAS AND GORENSTEIN TILED ORDERS

Hisaaki Fujita and Yosuke Sakai

Let D be a discrete valuation ring with a unique maximal ideal πD, and let Λ be a D-order. It is standard to reduce homological properties of Λ to those of the factor algebras $\Lambda/\pi\Lambda$ and such factor algebras are deserving of further study. (See [2].)

Let n be an integer with $n \geq 2$. In [1], we introduced an $n \times n$ $\mathbb{A}$-*full matrix algebra* over a field K, whose multiplication is determined by a *structure system* $\mathbb{A}$, that is, an n-tuple of $n \times n$ matrices with certain properties. $\mathbb{A}$-full matrix algebras are associative, basic, connected K-algebras. A prototype of $\mathbb{A}$-full matrix algebras is the class of factor algebras $\Lambda/\pi\Lambda$ of tiled D-orders Λ. Studying representation matrices of certain modules over $\mathbb{A}$-full matrix algebras, Frobenius $\mathbb{A}$-full matrix algebras are characterized by the shape of their structure systems $\mathbb{A}$. For a Gorenstein tiled D-order Λ, the factor algebra $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix algebra. In this paper we study the converse of this fact. Our main result is the following.

Theorem 1. (1) *For $2 \leq n \leq 7$, all Frobenius $n \times n$ $\mathbb{A}$-full matrix algebras are isomorphic to $\Lambda/\pi\Lambda$ for some Gorenstein tiled D-orders Λ. Moreover a list of them (up to isomorphism) is obtained.*

(2) *For each $n \geq 8$, there is a Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra having no corresponding Gorenstein tiled D-orders.*

1. $\mathbb{A}$-Full matrix algebras

We begin by recalling $\mathbb{A}$-full matrix algebras. Let K be a field and $n \geq 2$ an integer. Let $\mathbb{A} = (A_1, \ldots, A_n)$ be an n-tuple of $n \times n$ matrices $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \leq k \leq n)$ satisfying the following three conditions.

(A1) $a_{ij}^{(k)} a_{il}^{(j)} = a_{il}^{(k)} a_{kl}^{(j)}$ for all $i, j, k, l \in \{1, \ldots, n\}$,

(A2) $a_{kj}^{(k)} = a_{ik}^{(k)} = 1$ for all $i, j, k \in \{1, \ldots, n\}$, and

(A3) $a_{ii}^{(k)} = 0$ for all $i, k \in \{1, \ldots, n\}$ such that $i \neq k$.

Let $A = \bigoplus_{1 \leq i,j \leq n} K u_{ij}$ be a K-vector space with basis $\{u_{ij} \mid 1 \leq i, j \leq n\}$. Then we define multiplication of A by using $\mathbb{A}$, that is,

$$u_{ik} u_{lj} := \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Then $u_{11}, \ldots, u_{nn}$ are orthogonal primitive idempotents such that $u_{11} + \cdots + u_{nn} = 1_A$ an identity of A and $u_{ii} A u_{jj} \cong K$. Hence A is an associative, basic, connected K-algebra.

The detailed version of this paper will be submitted for publication elsewhere.

We call A an $n \times n$ $\mathbb{A}$-*full matrix algebra* with a *structure system* $\mathbb{A}$. We note that gl.dim$A = \infty$, because every entry of the Cartan matrix of A is 1.

In what follows, we assume that $a_{ij}^{(k)} = 0$ or 1 for all $1 \le i, k, j \le n$.

2. Tiled orders

Let D be a discrete valuation ring with a unique maximal ideal πD and $n \ge 2$ an integer. Let $\{\lambda_{ij} \mid 1 \le i, j \le n\}$ be a set of integers satisfying

$$\lambda_{ij} \ge 0, \quad \lambda_{ii} = 0, \quad \lambda_{ik} + \lambda_{kj} \ge \lambda_{ij}, \quad \text{and} \quad \lambda_{ij} + \lambda_{ji} > 0 \quad (\text{if } i \ne j)$$

for all $1 \le i, j, k \le n$. Then $\Lambda = (\pi^{\lambda_{ij}} D)$ is a subring of $\mathbb{M}_n(D)$, which we call an $n \times n$ *tiled D-order*.

Example 2. Let $\Lambda = (\pi^{\lambda_{ij}} D)$ be an $n \times n$ tiled D-order. Put $A := \Lambda/\pi\Lambda$, $K := D/\pi D$ and $u_{ij} := \pi^{\lambda_{ij}} e_{ij} + \pi\Lambda \in A$, where e_{ij}'s are the matrix units in $\mathbb{M}_n(D)$. Define $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \le k \le n)$ by

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } \lambda_{ik} + \lambda_{kj} = \lambda_{ij} \\ 0 & \text{otherwise,} \end{cases}$$

and set $\mathbb{A} := (A_1, \ldots, A_n)$. Then note that

$$u_{ik} u_{lj} = \begin{cases} a_{ij}^{(k)} u_{ij} & \text{if } k = l \\ 0 & \text{otherwise.} \end{cases}$$

Hence A is an $\mathbb{A}$-full matrix algebra.

3. Representation matrices

Let $A = \bigoplus_{1 \le i,j \le n} u_{ij} K$ be an $n \times n$ $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = (A_1, \ldots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \le k \le n)$. Let M be a right A-module with dimension type $\underline{\dim} M = (1, \ldots, 1)$. Then M has a K-basis $\{v_i \mid 1 \le i \le n\}$ such that $v_i u_{ii} = v_i$ for all $1 \le i \le n$. Hence there exists a matrix $S = (s_{ij}) \in \mathbb{M}_n(K)$ such that $v_i u_{ij} = s_{ij} v_j$ for all $1 \le i, j \le n$. We call S a *representation matrix* of M.

Proposition 3. *For each indecomposable projective right A-module $u_{ii}A$, $\underline{\dim} u_{ii}A = (1, \ldots, 1)$ and it has a representation matrix $(a_{ij}^{(k)})_{k,j}$, that is, an $n \times n$ matrix whose (k, j)-entry is $a_{ij}^{(k)}$. Moreover $u_{ii}A$ is isomorphic to an injective* $\mathrm{Hom}_K(Au_{ll}, K)$ *if and only if $a_{il}^{(k)} = 1$ for all $1 \le k \le n$.*

Example 4. Let A be an $\mathbb{A}$-full matrix algebra where

$$\mathbb{A} = \begin{pmatrix} 1&1&1&1 & 0&1&1&0 & 0&0&1&0 & 0&0&1&1 \\ 1&0&0&1 & 1&1&1&1 & 1&0&1&1 & 0&0&0&1 \\ 1&1&0&1 & 0&1&0&0 & 1&1&1&1 & 0&0&0&1 \\ 1&1&0&0 & 0&1&0&0 & 1&1&1&0 & 1&1&1&1 \end{pmatrix}.$$

Then representation matrices of $u_{11}A, \ldots, u_{44}A$ are given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Hence $u_{11}A, u_{22}A$ and $u_{44}A$ are injective but not $u_{33}A$.

4. Frobenius $\mathbb{A}$-full matrix algebras

By means of structure systems, we can characterize Frobenius $\mathbb{A}$-full matrix algebras. Let $A = \bigoplus_{1 \le i,j \le n} u_{ij}K$ be an $n \times n$ $\mathbb{A}$-full matrix algebra, where $\mathbb{A} = (A_1, \ldots, A_n)$ and $A_k = (a_{ij}^{(k)}) \in \mathbb{M}_n(K)$ $(1 \le k \le n)$.

Proposition 5. *The following are equivalent for an $\mathbb{A}$-full matrix algebra A.*

(1) *A is a Frobenius algebra.*

(2) *There exists a permutation σ of the set $\{1, \ldots, n\}$ such that $\sigma(i) \ne i$ for all $1 \le i \le n$ and that $a_{i\sigma(i)}^{(k)} = 1$ for all $1 \le i, k \le n$.*

In this case σ is a Nakayama permutation of A, that is, $\operatorname{soc}(u_{ii}A) \cong \operatorname{top}(u_{\sigma(i)\sigma(i)}A)$ *for all $1 \le i \le n$. Moreover, for all $1 \le i, k, j \le n$, $a_{ij}^{(k)} = a_{k\sigma(i)}^{(j)}$ holds.*

Using Proposition 5, we can find structure systems $\mathbb{A}$ which define Frobenius $\mathbb{A}$-full matrix algebras. Let σ be a permutation of the set $\{1, \ldots, n\}$ such that $\sigma(i) \ne i$ for all $1 \le i \le n$. Let T be the set of triples (i, k, j) of integers $1 \le i, k, j \le n$. Then we have a bijection

$$\varphi : T \to T, \ (i, k, j) \mapsto (k, j, \sigma(i)).$$

Decompose T into φ-orbits $\{T_\alpha\}_\alpha$, and put $I := \cup\{T_\alpha | \ (i, k, \sigma(i)) \in T_\alpha\}$, $Z := \cup\{T_\alpha | \ (i, k, i) \in T_\alpha, i \ne k\}$, and $X := \cup\{T_\alpha | \ T_\alpha \not\subset I \cup Z\}$. Then we have $T = I \cup Z \cup X$ (disjoint).

Proposition 6. (1) *Suppose that $\mathbb{A}$ is a Frobenius structure system with Nakayama permutation σ. Then there exists a φ-invariant subset Y of X such that*

$$a_{ij}^{(k)} = \begin{cases} 1 & \text{if } (i, k, j) \in I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Let Y be a φ-invariant subset of X, and define $\mathbb{A}(Y) = (a_{ij}^{(k)})$ by*

$$a_{ij}^{(k)} := \begin{cases} 1 & \text{if } (i, k, j) \in I \cup Y \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{A}(Y)$ is a Frobenius structure system whenever (A1) *holds for $\mathbb{A}(Y)$.*

(3) *For the empty subset $\emptyset$ of X, $\mathbb{A}(\emptyset)$ is a Frobenius structure system.*

5. φ-orbits for a cyclic permutation

In this section, we clarify the φ-orbits of T for a cyclic permutation $\sigma = (1 \ 2 \ \cdots \ n)$. First we count the number of φ-orbits of T.

Proposition 7. (1) *For a φ-orbit T_α of T, the number $|T_\alpha|$ of elements in T_α is $3n$ or n.*

(2) *T has a φ-orbit T_α with $|T_\alpha| = n$ if and only if n is not divisible by 3. In this case, T has a unique φ-orbit having n elements, which is contained in X.*

(3) *I has $n-1$ φ-orbits.*

(4) *Z has $n-2$ φ-orbits.*

(5) *If n is divisible by 3, then X has $(n-1)^2/3$ φ-orbits.*

(6) *If n is not divisible by 3, then X has $(n-2)(n-4)/3+1$ φ-orbits.*

Next we clarify the members of each φ-orbit of T.

Proposition 8. *Let T_α be a φ-orbit of T and put $T_\alpha^{(r)} := \{(i,k,j) \in T_\alpha \mid k = r\}$ for all $1 \le r \le n$.*

(1) *Suppose that $|T_\alpha| = 3n$. Then $|T_\alpha^{(r)}| = 3$ for each $r = 1, \ldots, n$. If $(i,1,j) \in T_\alpha$, then*

$$T_\alpha^{(1)} = \{(i,1,j),\ (\sigma^{-j+1}(1), 1, \sigma^{-j+2}(i)),\ (\sigma^{-i}(j), 1, \sigma^{-i+1}(1))\}.$$

(2) *Suppose that $|T_\alpha| = n$. Then $|T_\alpha^{(r)}| = 1$ for each $r = 1, \ldots, n$. If $n = 3t+1$ then $T_\alpha^{(1)} = \{(t+1, 1, 2t+2)\}$. If $n = 3t+2$ then $T_\alpha^{(1)} = \{(2t+2, 1, t+2)\}$.*

6. Minimal Frobenius structure systems

Let A be a Frobenius $\mathbb{A}$-full matrix algebra with Nakayama permutation σ. Then it follows from Proposition 6 (1) that $\mathbb{A}$ is determined by a φ_σ-invariant subset Y of X. We call $\mathbb{A}$ a *minimal Forbenius structure system* if Y is minimal among φ_σ-invariant subset of X which define Frobenius full matrix algebras. For a cyclic permutation, minimal Frobenius structure systems are determined by the following theorem.

Theorem 9. *Let n be an integer with $n \ge 4$, and let $\sigma = (1\ 2\ \cdots\ n)$ be a cyclic permutation. Then the following statements hold.*

(1) *Let n be even. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits contained in X.*

(2) *Let n be odd and $n = 2s+1$ for some s. Then the φ-invariant subsets defining minimal Frobenius structure systems are just φ-orbits X_β contained in X such that X_β does not contain any element of the form $(s+1, 1, k)$ for any k with $k \not\equiv s^2+1 \pmod{n}$.*

The following example illustrates Theorem 9.

Example 10. Let $n = 7$. Then X has 6 φ-orbits X_i $(1 \le i \le 6)$ such that

$$\begin{aligned}
X_1^{(1)} &= \{(4,1,3), (6,1,3), (6,1,5)\} \\
X_2^{(1)} &= \{(2,1,5), (3,1,7), (4,1,6)\} \\
X_3^{(1)} &= \{(2,1,6), (4,1,7), (3,1,5)\} \\
X_4^{(1)} &= \{(5,1,3), (5,1,4), (6,1,4)\} \\
X_5^{(1)} &= \{(2,1,4), (2,1,7), (5,1,7)\} \\
X_6^{(1)} &= \{(3,1,6)\}
\end{aligned}$$

Since $7 = 2 \cdot 3 + 1$ and $3^2 + 1 \equiv 3 \pmod{7}$, there are minimal Frobenius structure systems corresponding to X_1, X_4, X_5, X_6, but not to X_2, X_3.

7. Gorenstein tiled orders

A D-order Λ is *Gorenstein* if $\mathrm{Hom}_D(\Lambda, D)$ is projective as a right (or left) Λ-module. It is known that for an $n \times n$ tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$, Λ is Gorenstein if and only if there exists a permutation σ such that $\lambda_{ik} + \lambda_{k\sigma(i)} = \lambda_{i\sigma(i)}$ for all $1 \le i, k \le n$. (See [4].) Since $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda/\pi\Lambda} \Lambda/\pi\Lambda + 1$ and $\mathrm{id}_\Lambda \Lambda = \text{lattice-inj.dim}_\Lambda \Lambda + 1$, Λ is Gorenstein if and only if $\Lambda/\pi\Lambda$ is Frobenius. (We note that this fact follows from Proposition 5 in our context.) Hence some Frobenius $\mathbb{A}$-full matrix algebras are isomorphic to the factor algebras $\Lambda/\pi\Lambda$ for some Gorenstein tiled D-orders Λ. However the converse is not true.

Theorem 11. *For every integer $n \ge 8$, there exists a Frobenius $n \times n$ full matrix algebra which has no corresponding Gorenstein tiled orders.*

Proof. Let σ be a cyclic permutation $(1\ 2 \cdots\ n)$. Let X_β be a φ-orbit containing $(2, 1, 5)$. Then, since $n \ge 8$, it follows from Theorem 9 that X_β defines a minimal Frobenius structure system $\mathbb{A}$. Suppose that there exists a Gorenstein tiled D-order $\Lambda = (\pi^{\lambda_{ij}} D)$ such that $\Lambda/\pi\Lambda$ is a Frobenius $\mathbb{A}$-full matrix algebra. We may assume that $\lambda_{1j} = 0$ for all $1 \le j \le n$ by [3, Lemma 1.1]. Then we have $\lambda_{53} = 0$, which implies that $\lambda_{54} = 0$, so that $(3, 1, 5) \in X_\beta$, a contradiction. $\square$

In the ending part of the proof, we need a technical argument not included in this report, but the following example may be helpful to see the proof.

Example 12. When $n = 8$, the exponent matrix (λ_{ij}) of Gorenstein tiled D-orders $\Lambda = (\pi^{\lambda_{ij}} D)$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & 0 & a & b & c & d & c & b \\ b & 0 & 0 & b & r & s & s & r \\ c & 0 & x & 0 & c & s & t & s \\ d & 0 & y & y & 0 & d & s & s \\ c & 0 & z & w & z & 0 & c & r \\ b & 0 & y & w & w & y & 0 & b \\ a & 0 & x & y & z & y & x & 0 \end{pmatrix}$$

where $x := a - b,\ y := a - c,\ z := a - d,\ w := a + b - c - d,\ r := b + c - a,\ s := c + d - a$, and $t := 2c + d - b - a$.

8. The case of $2 \le n \le 7$

For each n $(2 \le n \le 7)$, we can verify that every Frobenius $n \times n$ $\mathbb{A}$-full matrix algebra A has a Gorenstein tiled D-order Λ such that $\Lambda/\pi\Lambda \cong A$. The following table shows that how many isomorphism classes of Frobenius $\mathbb{A}$-full matrix algebras are for each $n = 2, \dots, 7$.

n	no. of iso. classes
2	1
3	1
4	3
5	4
6	21
7	17

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Institute of Mathematics
University of Tsukuba
Tsukuba, Ibaraki 305-8571 JAPAN
E-mail: fujita@math.tsukuba.ac.jp, ysksakai@math.tsukuba.ac.jp