MORPHISMS REPRESENTED BY MONOMORPHISMS [1]

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Abstract. We answer a question posed by Auslander and Bridger. Every homomorphism of modules is projective-stably equivalent to an epimorphism but is not always to a monomorphism. We prove that a map is projective-stably equivalent to a monomorphism if and only if its kernel is torsionless, that is, a first syzygy. If it occurs although, there can be various monomorphisms that are projective-stably equivalent to a given map. But in this case there uniquely exists a "perfect" monomorphism to which a given map is projective-stably equivalent.

1 Introduction

Let R be a commutative noetherian ring. Linear maps $f : A \to B$ and $f' : A' \to B'$ of finite R-modules are said to be projective-stably equivalent (pse for short) if the following diagram is commutative

$$\begin{array}{ccc} A \oplus P' & \overset{\left(\begin{smallmatrix} f & s \\ t & u \end{smallmatrix}\right)}{\longrightarrow} & B \oplus Q' \\ \downarrow \cong & & \downarrow \cong \\ A' \oplus P & \overset{\left(\begin{smallmatrix} f' & s' \\ t' & u' \end{smallmatrix}\right)}{\longrightarrow} & B' \oplus Q \end{array}$$

with some projective modules P, Q, P', Q' and R-linear maps s, t, u, s', t', u'. We say a morphism f is represented by monomorphisms ("rbm" for short) if there exists a monomorphism that is pse to f.

For any homomorphism $f : A \to B$ of R-modules, $(\, f \quad \rho_B\,) : A \oplus P_B \to B$ is surjective with a projective cover $\rho_B : P_B \to B$. Thus every morphism is represented by epimorphisms. The choice of epimorphism is unique; if an epimorphism f' is pse to f, then two sequences $0 \to \operatorname{Ker} f' \to A' \overset{f'}{\to} B' \to 0$ and $0 \to \operatorname{Ker}(f\ \rho_B) \to A \oplus P_B \overset{(f\ \rho_B)}{\to} B \to 0$ becomes isomorphic after splitting off common projective summands.

The formal analogy to the representations by monomorphisms fails both in existence and in uniqueness. Every morphism is not always represented by monomorphisms (Example 1). Even if a morphism f is rbm, the choice of monomorphism is not unique;

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there may be two monomorphisms f' and f'' both pse to f and that $0 \to A' \xrightarrow{f'} B' \to \operatorname{Cok} f' \to 0$ and $0 \to A'' \xrightarrow{f''} B'' \to \operatorname{Cok} f'' \to 0$ are not isomorphic by splitting off common projective summands (Example 2).

The purpose of the paper is finding a condition of a given map to be rbm. Roughly speaking, our problem is to know when an exact sequence of modules

$$0 \to A \to B \xrightarrow{g} C \to 0$$

can be modified into an exact sequence

$$0 \to B \xrightarrow{f} C \to A' \to 0.$$

Of course the projective stabilization $\underline{\operatorname{mod} R}$ of $\operatorname{mod} R$ is not triangulated in general. So the obstruction for a given map to be rbm should be the obstruction for $\underline{\operatorname{mod} R}$ to be triangulated. Our first focus is an analogy to the homotopy category $\mathsf{K}(\operatorname{mod} R)$ of R-complexes. In [5, Theorem 2.6], the author showed a category equivalence between $\underline{\operatorname{mod} R}$

and a subcategory of $\mathsf{K}(\operatorname{mod} R)$. Due to this equivalence, we describe the obstruction of being rbm with a homology of a complex associated to the given map.

The problem was originally posed by Auslander and Bridger [1]. They proved that a map is rbm if and only if it is pse to a "perfect" monomorphism. An exact sequence of R-modules is called perfect if its R-dual is also exact. A perfect monomorphism refers to a monomorphism whose R-dual is an epimorphism. This is our next focal point. In the case that a map is rbm, the choice of a monomorphism is not unique, but then a perfect monomorphism pse to the given map is uniquely determined up to direct sum of projective modules. (Theorem 3.6.)

Looking at Theorem 3.6, we see that when a morphism is rbm, its pseudo-kernel is always the first syzygy of its pseudo-cokernel. So it is tempting to ask if the equivalent condition of rbm property is that the kernel is a submodule of a free module. This is our third point. Actually, we need to assume the total ring of fractions $Q(R)$ is Gorenstein: the condition is satisfied for instance if R is a domain.

> Theorem 4.8 : Suppose the total ring of fractions $Q(R)$ of a ring R is Gorenstein. A morphism f is rbm if and only if $\operatorname{Ker} f$ is a submodule of a free module.

Let us give easy examples:

Example 1 Set $R = k[[X,Y]]/(XY)$ with any field k, $g : R^2/\binom{X}{Y}R \to R/(X) \oplus R/(Y)$ with $g(\binom{a}{b} \bmod \binom{X}{Y}R) = (a \bmod (X), b \bmod (Y))$. Since $\operatorname{Ker} g \cong R/(X,Y)$ is not a first syzygy, g is not rbm due to Theorem 4.8.

Example 2 Set $R = k[[X,Y]]/(XY)$ with any field k, $f : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R$ with $f(\binom{a}{b} \bmod \binom{X}{Y}R) = \binom{Xa}{Yb} \bmod \binom{X^2}{Y^2}R$. The map f is not a monomorphism; $\operatorname{Ker} f \cong R/(X) \oplus R/(Y)$ is a first syzygy. By Theorem 4.8, f is rbm. In fact, let $f' : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R \oplus R^2$ be defined as $f'(\binom{a}{b} \bmod \binom{X}{Y}R) = (\binom{Xa}{Yb} \bmod \binom{X^2}{Y^2}R, \binom{Ya}{Xb})$. Obviously f'

is a monomorphism that is pse to f. On the other hand, $f'' : R^2/\binom{X}{Y}R \to R^2/\binom{X^2}{Y^2}R \oplus R^2$ with $f''(\binom{a}{b} \bmod \binom{X}{Y}R) = (\binom{Xa}{Yb} \bmod \binom{X^2}{Y^2}R, \binom{Y^2a}{X^2b})$ is also a monomorphism and pse to f. We have two exact sequences

$$\theta_f : 0 \to R^2/\tbinom{X}{Y}R \xrightarrow{f'} R^2/\tbinom{X^2}{Y^2}R \oplus R^2 \to R^2/\tbinom{X}{Y}R \oplus R^2/\tbinom{Y}{X}R \to 0,$$

and

$$\sigma : 0 \to R^2/\tbinom{X}{Y}R \xrightarrow{f''} R^2/\tbinom{X^2}{Y^2}R \oplus R^2 \to R^2/\tbinom{X}{Y^2}R \oplus R^2/\tbinom{Y}{X^2}R \to 0,$$

that are not isomorphic. We see θ_f is perfect but σ is not.

2 Stable module category and homotopy category

Throughout the paper, R is a commutative noetherian ring, By an "R-module" we mean a finitely generated R-module. For an R-module M, $\rho_M : P_M \to M$ denotes a projective cover of M.

Definition 2.1 *The projective stabilization* $\underline{\mathrm{mod}\, R}$ *is defined as follows.*

- *Each object of* $\underline{\mathrm{mod}\, R}$ *is an object of* mod R.
- *For objects* A, B *of* $\underline{\mathrm{mod}\, R}$*, a set of morphisms from* A *to* B *is* $\underline{\mathrm{Hom}}_R(A, B) = \mathrm{Hom_R}(A, B)/\mathcal{P}(A, B)$ *where* $\mathcal{P}(A, B) := \{f \in \mathrm{Hom_R}(A, B) \mid$ *f factors through some projective module*$\}$. *Each element is denoted as* $\underline{f} = f \bmod \mathcal{P}(A, B)$.

A morphism mod R *is called* a stable isomorphism *if* $\underline{f}$ *is an isomorphism in* $\underline{\mathrm{mod}\, R}$. *If two R-modules A and A' are isomorphic in* $\underline{\mathrm{mod}\, R}$*, we say A and A' are stably isomorphic and write* $A \overset{st}{\cong} A'$.

Definition 2.2 *Morphisms* $f : A \to B$ *and* $f' : A' \to B'$ *in* mod R *are said to be projective-stably equivalent (pse for short) and denoted as* $f \overset{st}{\cong} f'$ *if if there exist stable isomorphisms* $\alpha : A \to A'$ *and* $\beta : B \to B'$ *such that* $\underline{\beta \circ f} = \underline{f' \circ \alpha}$.

Let $\mathcal{L}$ be a full subcategory of $\mathsf{K}(\mathrm{mod}\, R)$ defined as

$$\mathcal{L} = \{F^\bullet \in \mathsf{K}(\mathrm{proj}\, R) \mid \mathrm{H}^i(F^\bullet) = 0\ (i < 0), \quad \mathrm{H}_j(F^*{}_\bullet) = 0\ (j \geq 0)\}.$$

Lemma 2.3 ([5] Proposition 2.3, Proposition 2.4)

1) *For* $A \in \underline{\mathrm{mod}\, R}$*, there exists* ${F_A}^\bullet \in \mathcal{L}$ *that satisfies*

$$\mathrm{H}^0(\tau_{\leq 0}{F_A}^\bullet) \overset{st}{\cong} A.$$

Such an ${F_A}^\bullet$ *is uniquely determined by A up to isomorphisms. We fix the notation* ${F_A}^\bullet$ *and call this a standard resolution of A.*

2) *For* $\underline{f} \in \underline{\mathrm{Hom}}_R(A, B)$*, there exists* $f^\bullet \in \mathrm{Hom}_{\mathsf{K}(\mathrm{proj\,R})}({F_A}^\bullet, {F_B}^\bullet)$ *that satisfies*

$$\underline{\mathrm{H}^0(\tau_{\leq 0} f^\bullet)} \cong \underline{f}.$$

Such an $f^\bullet$ *is uniquely determined by* $\underline{f}$ *up to isomorphisms, so we use the notation* $f^\bullet$ *to describe a chain map with this property for given* $\underline{f}$*.*

Theorem 2.4 ([5] Theorem 2.6) *The mapping* $A \mapsto {F_A}^\bullet$ *gives a functor from* $\underline{\mathrm{mod}\,R}$ *to* $\mathsf{K}(\mathrm{mod}\,R)$*, and this gives a category equivalence between* $\underline{\mathrm{mod}\,R}$ *and* $\mathcal{L}$*.*

For $f \in \mathrm{Hom}_{\mathrm{R}}(A, B)$, there exists a triangle

$$C(f)^{\bullet -1} \overset{{n_f}^\bullet}{\to} {F_A}^\bullet \overset{f^\bullet}{\to} {F_B}^\bullet \overset{{c_f}^\bullet}{\to} C(f)^\bullet. \tag{2.1}$$

In general, $C(f)^\bullet$ does not belong to $\mathcal{L}$ but it satisfies the following:

$$\mathrm{H}^i(C(f)^\bullet) = 0 \ (i < -1), \quad \mathrm{H}_j(C(f)^*{}_\bullet) = 0 \ (j > -1).$$

Definition and Lemma 2.5 ([5], Definition and Lemma 3.1) *As objects of* $\underline{\mathrm{mod}\,R}$*,* $\underline{\mathrm{Ker}}\ \underline{f} := \mathrm{H}^{-1}(\tau_{\leq -1} C(f)^\bullet)$ *and* $\underline{\mathrm{Cok}}\ \underline{f} := \mathrm{H}^0(\tau_{\leq 0} C(f)^\bullet)$ *are uniquely determined by* $\underline{f}$*, up to isomorphisms. We call these the pseudo-kernel and the pseudo-cokernel of* $\underline{f}$*.*

For a given map $f : A \to B$, from (2.1), we have an exact sequence

$$0 \to \underline{\mathrm{Ker}}\ \underline{f} \to A \oplus P \overset{(f\ \rho)}{\to} B \to 0 \tag{2.2}$$

with some projective module P. This characterizes the pseudo-kernel.

Lemma 2.6 *For a given* $f \in \mathrm{Hom}_{\mathrm{R}}(A, B)$*, suppose* $A \oplus P' \overset{(f\ p')}{\to} B$ *is epimorphism with projective module* P'*. Then* $\mathrm{Ker}\,(f\ p') \overset{st}{\cong} \underline{\mathrm{Ker} f}$ *and the sequence*

$$0 \quad \to \quad \mathrm{Ker}\,(f\ p') \quad \to \quad A \oplus P' \quad \overset{(f\ p')}{\to} \quad B \quad \to \quad 0$$

is isomorphic to 2.2 after splitting off some aplit exact sequence of projective modules.

Lemma 2.7 ([5] Lemma 3.6)

1) *There is an exact sequence*

$$0 \to \mathrm{Ker}\, f \to \underline{\mathrm{Ker}}\ f \to \Omega^1_R(\mathrm{Cok}\ \ f) \to 0.$$

2) *There is an exact sequence*

$$0 \to L \to \underline{\mathrm{Cok}}\ \underline{f} \to \mathrm{Cok}\ \ f \to 0$$

such that $\Omega^1_R(L)$ *is the surjective image of* $\mathrm{Ker}\, f$*.*

3 Representation by monomorphisms and perfect exact sequences

Definition 3.1 *A morphism $f : A \to B$ in* $\operatorname{mod} R$ *is said to be* represented by monomorphisms *(rbm for short) if some monomorphism $f' : A' \to B'$ in* $\operatorname{mod} R$ *is pse to f, that is, there exist stable isomorphisms $\alpha : A \to A'$ and $\beta : B \to B'$ such that $\underline{\beta \circ f} = \underline{f' \circ \alpha}$.*

Each morphism is not always rbm. It was Auslander and Bridger who first defined and studied "represented by monomorphisms" property.

Theorem 3.2 (Auslander-Bridger) *The following are equivalent for a morphism $f : A \to B$ in* $\operatorname{mod} R$.

1) *There exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$.*

2) *There exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$, and f'^* is an epimorphism.*

3) $\mathrm{Hom_R}(B, I) \to \mathrm{Hom_R}(A, I)$ *is surjective if I is an injective module.*

The condition 1) of Theorem 3.2 turns out to be equivalent to the rbm condition.

Lemma 3.3 *For a morphism $f : A \to B$ in* $\operatorname{mod} R$*, f is rbm if and only if there exists a monomorphism $f' : A \to B \oplus P$ with a projective module P such that $f = s \circ f'$ via some split epimorphism $s : B \oplus P \to B$.*

The most remarkable point in Auslander-Bridger's Theorem is that being rbm is equivalent to being represented by "perfect monomorphisms" whose R-dual is an epimorphism.

Definition 3.4 *An exact sequence $0 \to A \to B \to C \to 0$ of R-modules is called a perfect exact sequence or to be perfectly exact if its R-dual $0 \to \mathrm{Hom_R}(C, R) \to \mathrm{Hom_R}(B, R) \to \mathrm{Hom_R}(A, R) \to 0$ is also exact. A monomorphism f is called a perfect monomorphism if* $\mathrm{Hom_R}(f, R)$ *is an epimorphism.*

Proposition 3.5 ([5] Lemma 2.7) *The following are equivalent for an exact sequence*

$$\theta : 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

1) *θ is perfectly exact.*

2) $0 \to {F_A}^{\bullet} \xrightarrow{f^{\bullet}} {F_B}^{\bullet} \xrightarrow{g^{\bullet}} {F_C}^{\bullet} \to 0$ *is exact.*

3) $F_C{}^{\bullet -1} \to F_A{}^{\bullet} \xrightarrow{f^{\bullet}} F_B{}^{\bullet} \xrightarrow{g^{\bullet}} F_C{}^{\bullet}$ *is a distinguished triangle in* $\mathsf{K}(\operatorname{mod} R)$.

For a morphism $f : A \to B$, $A \oplus P_B \xrightarrow{(f\ \rho_B)} B$ is an epimorphism with a projective cover $\rho_B : P_B \to B$. Thus each morphism is represented by epimorphisms. And the choice of the representing epimorphism is unique up to direct sum of projective modules, as we have seen in Lemma 2.6.

Unlikely, we already know an example of a morphism that is not rbm. And moreover, even if a given map is represented by a monomorphism, there would be another representing monomorphism. (Example 1 and Example 2.)

However, uniqueness theorem is obtained in this way. Due to Theorem 3.2, a morphism is rbm if and only if it is represented by a perfect monomorphism. And if this is the case, the representing perfect monomorphism is uniquely determined up to direct sum of projective modules.

Theorem 3.6 *Let* $f : A \to B$ *be a morphism in* $\operatorname{mod} R$. *Then* f *is rbm if and only if* $\mathrm{H}^{-1}(C(f)^{\bullet})$ *vanishes. If this is the case, we have the following:*

1) *We have a perfect exact sequence*

$$\theta_f : 0 \to A \xrightarrow{\binom{f}{\epsilon}} B \oplus F_A{}^1 \xrightarrow{(c^f\ \pi)} \underline{\mathrm{Cok} f} \to 0.$$

2) *For any exact sequence of the form*

$$\sigma : 0 \to A \xrightarrow{\binom{f}{q}} B \oplus P' \xrightarrow{(g\ p)} C \to 0$$

with some projective module P', *there is a commutative diagram*

$$\begin{array}{ccccccccccc} \theta_f : & 0 & \to & A & \xrightarrow{\binom{f}{\epsilon}} & B \oplus F_A{}^1 & \xrightarrow{(c^f\ \pi)} & \underline{\mathrm{Cok} f} & \to & 0 \\ & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ \sigma : & 0 & \to & A & \xrightarrow{\binom{f}{q}} & B \oplus P' & \xrightarrow{(g\ p)} & C & \to & 0 \end{array}$$

where α *and* β *are stable isomorphisms.*

3) *There is an exact sequence with some projective module* Q *and* Q'

$$0 \to Q' \to \underline{\mathrm{Cok} f} \oplus Q \xrightarrow{(\gamma\ \rho)} C \to 0.$$

In other words, $\underline{\mathrm{Ker}\gamma}$ *is projective.*

4) *If* σ *is also perfectly exact, then* σ *is isomorphic to* θ_f *up to direct sum of split exact sequences of projective modules.*

proof. We have a triangle

$$F_A{}^\bullet \overset{f^\bullet}{\to} F_B{}^\bullet \to C(f)^\bullet \overset{n_f{}^{\bullet+1}}{\to} F_A{}^{\bullet+1} \tag{3.3}$$

which induces a term-wise exact sequence of complexes in $\mathsf{C}(\operatorname{proj} R)$

$$0 \to F_A{}^\bullet \to C(n_f)^\bullet \to C(f)^\bullet \to 0 \tag{3.4}$$

Applying $\tau_{\le 0}$ to the diagram above and taking homology, we get the following exact sequence of modules:

$$\theta_f: \ 0 \to \mathrm{H}^{-1}(C(f)^\bullet) \to A \overset{\binom{f}{\epsilon}}{\to} B \oplus F_A{}^1 \overset{(c^f\ \pi)}{\to} \underline{\mathrm{Cok} f} \to 0 \tag{3.5}$$

Suppose that $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. Then $C(f)^\bullet \cong F_{\underline{\mathrm{Cok} f}}$, and the exact sequence 3.4 shows that θ_f is perfectly exact.

Conversely, suppose that f is rbm; there is an exact sequence

$$\sigma : 0 \to A \overset{\binom{f}{q}}{\to} B \oplus P' \overset{(g\ p)}{\to} C \to 0.$$

The maps $\tilde{f} = \binom{f}{q}$ and $\tilde{g} = (g\ p)$ produce the similar diagram as (3.3) :

$$\begin{array}{ccccccc} F_A{}^\bullet & \overset{\tilde{f}^\bullet}{\to} & F_{B\oplus P'}{}^\bullet & \to & C(\tilde{f})^\bullet & \to & F_A{}^{\bullet+1} \\ \downarrow{\scriptstyle \tilde{\alpha}^\bullet} & & \| & & \downarrow{\scriptstyle \tilde{\gamma}^\bullet} & & \downarrow{\scriptstyle \tilde{\alpha}^{\bullet+1}} \\ C(\tilde{g})^{\bullet-1} & \to & F_{B\oplus P'}{}^\bullet & \overset{\tilde{g}^\bullet}{\to} & F_C{}^\bullet & \to & C(\tilde{g})^\bullet \end{array} \tag{3.6}$$

Since $A \overset{st}{\cong} \underline{\mathrm{Ker}\tilde{g}}$, $\tau_{\le 0}\tilde{\alpha}^\bullet = 0$ is an isomorphism, equivalently $\tau_{\le -1}C(\tilde{\alpha})^\bullet = 0$ hence $\tau_{\le -2}C(\tilde{\gamma})^\bullet = 0$. From the long exact sequence of homology groups $\mathrm{H}^{-2}(C(\tilde{\gamma})^\bullet) \to \mathrm{H}^{-1}(C(\tilde{f})^\bullet) \to \mathrm{H}^{-1}(F_C{}^\bullet)$, we get $\mathrm{H}^{-1}(C(\tilde{f})^\bullet) = 0$. Obviously, $\mathrm{H}^{-1}(C(\tilde{f})^\bullet) \cong \mathrm{H}^{-1}(C(f)^\bullet)$ hence $\mathrm{H}^{-1}(C(f)^\bullet) = 0$. Now it remains to prove 2) - 4) in the case $\mathrm{H}^{-1}(C(f)^\bullet) = 0$.

2) Applying $\tau_{\le 0}$ to the diagram (3.6) and taking homology, we get the following diagram with exact rows:

$$\begin{array}{rccccccccc} \theta_{\tilde{f}}: & 0 & \to & A & \overset{\left(\begin{smallmatrix} f \\ q \\ \epsilon \end{smallmatrix}\right)}{\to} & B \oplus P' \oplus F_A{}^1 & \to & \underline{\mathrm{Cok}\binom{f}{q}} & \to & 0 \\ & & & \downarrow{\scriptstyle \tilde{\alpha}'} & & \downarrow{\scriptstyle \tilde{\beta}'} & & \downarrow{\scriptstyle \tilde{\gamma}'} & & \\ \sigma_{\tilde{g}}: & 0 & \to & \underline{\mathrm{Ker}(g\ p)} & \to & B \oplus P' \oplus P_C & \overset{(g\ p\ \rho_C)}{\to} & C & \to & 0. \end{array}$$

Notice that $\tilde{\alpha}'$ and $\tilde{\beta}'$ are stable isomorphisms. The upper row is a direct sum of θ_f and a trivial complex, and the lower row is that of σ and a trivial complex. Splitting off trivial complexes we get a desired diagram:

$$\begin{array}{ccccccccc} 0 & \to & A & \overset{\binom{f}{\epsilon}}{\to} & B \oplus F_A{}^1 & \overset{(c^f\ \pi)}{\to} & \underline{\mathrm{Cok} f} & \to & 0 \\ & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \beta'} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \to & A & \overset{\binom{f}{q}}{\to} & B \oplus P' & \overset{(g\ p)}{\to} & C & \to & 0 \end{array}$$

3) As we see above, $\tilde{\gamma}^{\bullet} : C(\tilde{f})^{\bullet} = F_{\underline{\mathrm{Cok}}\tilde{f}}{}^{\bullet} \to F_C{}^{\bullet}$ has $\tau_{\le -2}C(\tilde{\gamma})^{\bullet} = 0$. We may consider $\tilde{\gamma}^{\bullet}$ as $\tilde{\gamma}_i = \mathrm{id} \quad (i \le -1)$ hence $Q' = \underline{\mathrm{Ker}}\tilde{\gamma}$ is projective;

$$0 \to Q' \to \underline{\mathrm{Cok}}\tilde{f} \oplus P_C \stackrel{(\tilde{\gamma}'\ \rho_C)}{\to} C \to 0.$$

Since $\underline{\mathrm{Cok}}\tilde{f} \stackrel{st}{\cong} \underline{\mathrm{Cok}}f$ and $\tilde{\gamma}' \stackrel{st}{\cong} \gamma$, the above sequence is the desired sequence $0 \to Q' \to \underline{\mathrm{Cok}}f \oplus Q \to C \to 0$ with some projective module Q.

4) Suppose σ is perfect. From Proposition 3.5, $F_C{}^{\bullet -1} \to F_A{}^{\bullet} \xrightarrow{f^{\bullet}} F_B{}^{\bullet} \xrightarrow{g^{\bullet}} F_C{}^{\bullet}$ is a distinguished triangle, and $F_C{}^{\bullet} \cong C(f)^{\bullet}$, hence the induced sequence σ is isomorphic to θ_f. (q.e.d.)

4 Representation by monomorphisms and torsionless modules.

In the previous section, we see that a given map f is represented by monomorphisms if and only if $\mathrm{H}^{-1}(C(f)^{\bullet}) = 0$. If this is the case, $\underline{\mathrm{Ker}}f = \mathrm{Cok}\, d_{C(f)}{}^{-2}$ is the first syzygy of $\underline{\mathrm{Cok}}f = \mathrm{Cok}\, d_{C(f)}{}^{-1}$. So it is natural to ask the converse: Is a given map f represented by monomorphisms if $\underline{\mathrm{Ker}}f$ is a first sygyzy? This section deals with the problem. As a conclusion, the answer is yes if the total ring of fractions $Q(R)$ of R is Gorenstein. What is more, if $Q(R)$ is Gorenstein, instead of a pseudo-kernel, we can use a (usual) kernel to describe rbm condition.

The next is well known. See [1] and [4] for the proof.

Definition and Lemma 4.1 *The following are equivalent for an R-module M.*

1) *The natural map $\phi : M \to M^{**}$ is a monomorphism.*

2) $\mathrm{Ext}^1_{\mathrm{R}}(\mathrm{Tr}\, M, R) = 0$

3) *M is a first syzygy; there exists a monomorphism from M to a projective module.*

If M satisfies these conditions, M is said to be torsionless. [1]

To solve our problem, the special kind of maps is a key. For $M \in \mathrm{mod}\ R$, consider a module $J^2M = \mathrm{Tr}\,\Omega^1_R \mathrm{Tr}\,\Omega^1_R M$. Since $\mathrm{Tr}\, J^2M$ is a first sygyzy, we have $\mathrm{Ext}^1_{\mathrm{R}}(J^2M, R) = 0$, which means $\mathrm{H}_{-1}(F_{J^2M}{}^{*}{}_{\bullet}) = 0$ and $\tau_{\ge -2}F_{J^2M}{}^{*}{}_{\bullet}$is a projective resolution of $\mathrm{Tr}\,\Omega^1_R M = \mathrm{Cok}\,(d_{F_{J^2M}}{}^{-2})^{*} = \mathrm{Cok}\,(d_{F_M}{}^{-2})^{*}$. The identity map on $\mathrm{Tr}\,\Omega^1_R M$ induces a chain map $(F_M)^{*}_{\bullet} \to (F_{J^2M})^{*}_{\bullet}$ and its R-dual $\psi_M{}^{\bullet} : F_{J^2M}{}^{\bullet} \to F_M{}^{\bullet}$ subsequently.

Lemma 4.2 *The map $\psi_M : J^2M \to M$ is rbm if and only if an R-module M has* $(\mathrm{Ext}^1_{\mathrm{R}}(M, R))^{*} = 0$.

[1] In [1], Auslander and Bridger use the term "1-torsion free" for "torsionless". Usually a module M is called torsion-free if the natural map $M \to M \otimes Q(R)$ is injective.

proof. From Theorem 3.6, ψ_M is rbm if and only if $\mathrm{H}^{-1}(C(\psi_M)^\bullet) = 0$. By definition, ψ_M^{-1} and ψ_M^{-2} are identity maps hence ψ_M^i are identity maps for $i \leq -1$. We may assume $\tau_{\leq -2}C(\psi_M)^\bullet = 0$, which implies $\mathrm{H}^{-1}(C(\psi_M)^\bullet) = \operatorname{Ker} d_{C(\psi_M)}{}^{-1}$. As $\tau_{\geq -1}C(\psi_M)_\bullet{}^*$ is a projective resolution of $\operatorname{Cok}(d_{\psi_M}{}^{-1})^* \cong \mathrm{H}_{-1}(C(\psi_M)^*_\bullet)$, we get $\mathrm{H}^{-1}(C(\psi_M)^\bullet) \cong (\mathrm{H}_{-1}(C(\psi_M)^*_\bullet))^*$. A triangle $F_{J^2M}{}^\bullet \overset{\psi_M{}^\bullet}{\to} F_M{}^\bullet \to C(\psi_M)^\bullet \to F_{J^2M}{}^{\bullet+1}$ induces an R-dual triangle $F_{J^2M}{}^*_{\bullet+1} \to C(\psi_M)^*_\bullet \to F_M{}^*_\bullet \to F_{J^2M}{}^*_\bullet$ which produces an exact sequence of modules

$$0 \to \mathrm{H}_{-1}(C(\psi_M)^*_\bullet) \to \mathrm{H}_{-1}(F_M{}^*_\bullet) \to \mathrm{H}_{-1}(F_{J^2M}{}^*_\bullet) \to 0.$$

As we see in the discussion above, $\mathrm{H}_{-1}(F_{J^2M}{}^*_\bullet) = 0$. Hence $\mathrm{H}_{-1}(C(\psi_M)^*_\bullet) \cong \mathrm{H}_{-1}(F_M{}^*_\bullet) = \mathrm{Ext}^1_{\mathrm{R}}(M, R)$, and we get $\mathrm{H}^{-1}(C(\psi_M)^\bullet) \cong (\mathrm{Ext}^1_{\mathrm{R}}(M, R))^*$. (q.e.d.)

The above result is generalized as follows:

Lemma 4.3 *Let $f : A \to B$ be a morphism in* $\operatorname{mod} R$. *Suppose* $(\mathrm{Ext}^1_{\mathrm{R}}(B, R))^* = 0$. *If* $\underline{\operatorname{Ker} f}$ *is projective, then f is rbm.*

proof. We may assume $\tau_{\leq -2}C(f)^\bullet = 0$. Similarly as in the proof of Lemma 4.2, we have $\mathrm{H}^{-1}(C(f)^\bullet) \cong (\mathrm{H}_{-1}(C(f)^*_\bullet))^*$. Since $\underline{\operatorname{Ker} f}$ is projective, f induces a stable isomorphism $J^2A \overset{st}{\cong} J^2B$, and via this stable isomorphism, ψ_B is projective stably equivalent to $f \circ \psi_A$, equivalently $\psi_B{}^\bullet \cong f^\bullet \circ \psi_A{}^\bullet$ in $\mathsf{K}(\operatorname{mod} R)$. We have a triangle

$$C(\psi_A)^\bullet \to C(\psi_B)^\bullet \to C(f)^\bullet \to C(\psi_A)^{\bullet+1}$$

and its R-dual

$$C(\psi_A)^*{}_{\bullet+1} \to C(f)^*{}_\bullet \to C(\psi_B)^*{}_\bullet \to C(\psi_A)^*{}_\bullet$$

which induce an exact sequence of modules

$$0 \to \mathrm{H}_{-1}(C(f)^*_\bullet) \to \mathrm{H}_{-1}(C(\psi_B)^*_\bullet)$$

Note that $\mathrm{H}_{-1}(C(\psi_B)^*)_\bullet) = \mathrm{Ext}^1_{\mathrm{R}}(B, R)$. The assumption $(\mathrm{Ext}^1_{\mathrm{R}}(B, R))^* = 0$ equivalently says $\mathrm{Ext}^1_{\mathrm{R}}(B, R)_{\mathbf{p}} = 0$ for any associated prime ideal $\mathbf{p}$ of R. A submodule has the same property; $\mathrm{H}_{-1}(C(f)^*_\bullet)_{\mathbf{p}} = 0$ for any associated prime ideal $\mathbf{p}$ of R therefore $(\mathrm{H}_{-1}(C(f)^*_\bullet))^* = 0$. (q.e.d.)

Proposition 4.4 *Let $f : A \to B$ a morphism of* $\operatorname{mod} R$. *Suppose* $(\mathrm{Ext}^1_{\mathrm{R}}(B, R))^* = 0$. *Then f is rbm if and only if* $\underline{\operatorname{Ker} f}$ *is torsionless.*

proof. We already get the "only if" part and have only to show the "if" part. Adding a projective cover of B to f, we get an exact sequence

$$\sigma_f : 0 \to \underline{\operatorname{Ker} f} \overset{\binom{n^f}{q}}{\to} A \oplus P_B \overset{(f\ \rho_B)}{\to} B \to 0.$$

Due to Theorem 3.6, we have a perfect exact sequence θ_{n^f}, because n^f is rbm:

$$\begin{array}{cccccccccc} \theta_{n^f}: & 0 & \to & \underline{\mathrm{Ker} f} & \stackrel{\binom{n^f}{\epsilon}}{\to} & A \oplus F^1_{\underline{\mathrm{Ker} f}} & \stackrel{\left(c^{n^f}\ \pi\right)}{\to} & \underline{\mathrm{Cok}}\ \underline{n^f} & \to & 0 \\ & & & \downarrow{\scriptstyle st}\ \cong & & \downarrow{\scriptstyle st}\ \cong & & \downarrow{\scriptstyle \omega_f} & & \\ \sigma_f & 0 & \to & \underline{\mathrm{Ker} f} & \stackrel{\binom{n^f}{q}}{\to} & A \oplus P_B & \stackrel{(f\ \rho_B)}{\to} & B & \to & 0. \end{array}$$

From Theorem 3.6 3), we know $\underline{\mathrm{Ker}\omega_f}$ is projective. With the assumption $(\mathrm{Ext}^1_{\mathrm{R}}(B, R))^* = 0$, we can apply Lemma 4.3 and get that ω_f is rbm. From the equation $\underline{f} = \underline{\omega_f} \circ \underline{c^{n^f}}$, f is rbm if c^{n^f} is rbm. Since

$$F_{\underline{\mathrm{Ker} f}}{}^{\bullet} \stackrel{n^{f\bullet}}{\to} F_A{}^{\bullet} \stackrel{c^{n^f\bullet}}{\to} C(n_f)^{\bullet} \to F_{\underline{\mathrm{Ker} f}}{}^{\bullet+1}$$

is a triangle, $C(c^{n^f})^{\bullet} \cong F^{\bullet+1}_{\underline{\mathrm{Ker} f}}$; $\mathrm{H}^{-1}(C(c^{n^f})^{\bullet}) \cong \mathrm{H}^0(F_{\underline{\mathrm{Ker} f}}{}^{\bullet}) \cong \mathrm{Ext}^1_{\mathrm{R}}(\mathrm{Tr}\,\underline{\mathrm{Ker} f}, R)$. Hence c^{n^f} is rbm if and only if $\underline{\mathrm{Ker} f}$ is torsionless. (q.e.d.)

Lemma 4.5 *Let the sequence of R-modules $0 \to A \stackrel{f}{\to} B \stackrel{g}{\to} C \to 0$ be exact. Suppose $(\mathrm{Ext}^1_{\mathrm{R}}(C, R))^* = 0$. If A and C are torsionless, then so is B.*

proof. From the assumption, $A \cong \underline{\mathrm{Ker} g}$ is torsionless. Due to Proposition 4.4, g is rbm; there exists an exact sequence

$$\theta_g : 0 \to B \stackrel{\binom{g}{\epsilon}}{\to} C \oplus Q \to \underline{\mathrm{Cok} g} \to 0$$

with a projective module Q and a map $\epsilon : B \to Q$. Since C is a submodule of some projective module, so is B. (q.e.d.)

Proposition 4.6 *The following are equivalent for a noetherian ring R.*

1) *$Q(R)$ is Gorenstein.*

2) *$Q(R)$ is Gorenstein of dimension zero.*

3) $(\mathrm{Ext}^1_{\mathrm{R}}(M, R))^* = 0$ *for each $M \in \mathrm{mod}\, R$.*

4) *Ψ_M is rbm for each $M \in \mathrm{mod}\, R$.*

If R is a local ring with the maximal ideal $\mathbf{m}$, *the above conditions are also equivalent to the following.*

5) *$\Psi_{R/\mathbf{m}}$ is rbm.*

proof. As $Q(R)$ is always of dimension zero, we get 1) $\Leftrightarrow$ 2).

3) $\Leftrightarrow$ 4) is already shown in Lemma 4.2.

4) $\Rightarrow$ 5) is obvious.

5) $\Rightarrow$ 1). The condition 5) is equivalent to $\mathrm{Ext}^1_{\mathrm{R}}(R/\mathbf{m}, R)\otimes Q(R) \cong \mathrm{Ext}^1_{\mathrm{Q(R)}}(R/\mathbf{m}\otimes Q(R), Q(R)) = 0$, which means $Q(R)$ is Gorenstein. (q.e.d.)

In the case $Q(R)$ is Gorenstein, every morphism in $\mathrm{mod}\, R$ satisfies the hypotheses of Proposition 4.4 and Lemma 4.5. Thus with the condition $Q(R)$ is Gorenstein, when discussing rbm property, we can deal with normal kernel as well as pseudo-kernel.

Proposition 4.7 *Suppose $Q(R)$ is Gorenstein. For a given morphism f,* $\mathrm{Ker}\, f$ *is torsionless if and only if* $\underline{\mathrm{Ker} f}$ *is torsionless.*

proof. From Lemma 2.7, there is an exact sequence $0 \to \mathrm{Ker}\, f \to \underline{\mathrm{Ker}}\ f \to \Omega^1_R(\mathrm{Cok}\, f) \to 0$. So the "if" part is obvious, and the "only if" part comes from Lemma 4.5. (q.e.d.)

Theorem 4.8 *Suppose $Q(R)$ is Gorenstein. The following are equivalent for a morphism $f : A \to B$ in* $\mathrm{mod}\, R$.

1) *f is rbm.*

2) $\mathrm{Ker}\, f$ *is torsionless.*

3) $\underline{\mathrm{Ker} f}$ *is torsionless.*

4) $\mathrm{H}^{-1}(C(f)^\bullet) = 0$.

5) $\Omega^1_R(\underline{\mathrm{Cok} f}) \overset{st}{\cong} \underline{\mathrm{Ker} f}$.

6) *There exists f' such that $f' \overset{st}{\cong} f$ and* $\mathrm{Ker}\, f'$ *is torsionless.*

7) *For any f' with $f' \overset{st}{\cong} f$,* $\mathrm{Ker}\, f'$ *is torsionless.*

proof. Implications 5) $\Rightarrow$ 3), 7) $\Rightarrow$ 2) and 7) $\Rightarrow$ 6) are obvious. We already showed 1) $\Leftrightarrow$ 4) in Theorem 3.6, 1) $\Leftrightarrow$ 3) in Proposition **??**, and 3)$\Leftrightarrow$ 2) in Corollary 4.7. Implications 3) $\Rightarrow$ 7) and 6) $\Rightarrow$ 3) are obtained from "if" and "only if " part of Corollary 4.7 respectively.

4) $\Rightarrow$ 5). It comes directly from $\mathrm{Cok}\, {d_{C(f)}}^0 = \underline{\mathrm{Ker} f}$ and $\mathrm{Cok}\, {d_{C(f)}}^{-1} = \underline{\mathrm{Cok} f}$. (q.e.d.)

Remark 4.9 *Takashima gives an easy proof for Theorem 4.8 using the torsion theory [7].*

Corollary 4.10 *The following are equivalent for a noetherian ring R.*

1) *$Q(R)$ is Gorenstein.*

2) *Every morphism with torsionless kernel is rbm.*

proof.

1) $\Rightarrow$ 2). It comes directly from Theorem 4.8.

2) $\Rightarrow$ 1). For every $M \in \operatorname{mod} R$, $\operatorname{Ker}\psi_M$ is torsionless. Because $\underline{\operatorname{Ker}\psi_M}$ is projective and $\operatorname{Ker}\psi_M$ is a submodule of $\underline{\operatorname{Ker}\psi_M}$ from Lemma 2.7 1). So if 2) holds, ψ_M is rbm for any $M \in \operatorname{mod} R$, which implies 1) from Proposition 4.6. (q.e.d.)

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