SOME TOPICS ON DERIVED EQUIVALENT BLOCKS OF FINITE GROUPS

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1. Introduction

Let $G$ be a finite group. Let $k$ be an algebraically closed field of characteristic $\ell > 0$. We denote the principal block of $kG$ by $B_0(G)$.

We say that two finite groups $G$ and $H$ have the same $\ell$-local structure if $G$ and $H$ have a common Sylow $\ell$-subgroup $P$ such that whenever $Q_1$ and $Q_2$ are subgroups of $P$ and $f : Q_1 \to Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$.

There is a well known conjecture due to Broué.

**Conjecture 1.1**(Broué [1, 2]). Let $G$ and $H$ be finite groups having the same $\ell$-local structure with common Sylow $\ell$-subgroup $P$. If $P$ is abelian then the principal blocks of $G$ and $H$ would be derived equivalent. If $P$ is not abelian, then there is a counterexample to this conjecture. However, there are some examples that $P$ is not abelian and there is a derived equivalence between the principal blocks of $G$ and $H$. We will give such examples in §3.

2. General theory

In this section, let $G$ and $H$ be finite groups having the same $\ell$-local structure with common Sylow $\ell$-subgroup $P$. We say that a complex of $(B_0(G), B_0(H))$-bimodules is splendid if each indecomposable summand of each term of the complex is a direct summand of a module of the form $kG \otimes_{kQ} kH$ for a subgroup $Q$ of $P$.

**Definition 2.1.** Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a splendid stable equivalence if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{**} \cong B_0(G) \oplus Z_1, \quad X^{**} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2$$

where $Z_1$ and $Z_2$ are homotopy equivalent to complexes of projective bimodules.

**Definition 2.2.** Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a splendid equivalence if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{**} \cong B_0(G) \oplus Z_1, \quad X^{**} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2$$

where $Z_1$ and $Z_2$ are homotopy equivalent to 0. The complex $X^\bullet$ is called a splendid tilting complex.

The detailed version of this paper will be submitted for publication elsewhere.
By the definition, splendid equivalences induce derived equivalences and homotopy equivalences.

**Theorem 2.1** (Rouquier [12]). Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. Then the following are equivalent.

1. The complex $X^\bullet$ induces a splendid stable equivalence between $B_0(G)$ and $B_0(H)$.
2. For every non-trivial subgroup $Q$ of $P$, the complex $X^\bullet(\Delta(Q))$ induces a splendid equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$, where $\Delta(Q)$ is a diagonal subgroup and

$$X(Q) = X^{\Delta(Q)} / \sum_{R<Q} \text{Tr}_R^Q X^{\Delta(R)}.$$ 

In our example in §3 we will use the following method when we prove splendid equivalences.

**(Step 1)** Construct a splendid tilting complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for every non-trivial subgroup $Q$ of $P$.

**(Step 2)** Construct a splendid stable equivalence $F$ from $B_0(G)$ to $B_0(H)$ by gluing the splendid tilting complexes obtained in Step 1 (by using the above theorem).

**(Step 3)** Calculate $F(S)$ for the simple $B_0(G)$-modules.

**(Step 4)** Lift the stable equivalence in Step 2 to a splendid equivalence by looking at the modules calculated in Step 3.

### 3. General linear groups and unitary groups

Let $q$ be a power of a prime. Assume that $\ell$ is odd and $\ell^e$ divides $q + 1$ but $\ell^{e+1}$ does not divide $q + 1$ for some $e > 0$. Under this condition, we consider representations of the general linear group $GL(n, q^2)$ and the unitary group $GU(n, q^2)$ for small $n$. Note that if $\ell > n$ then the principal $\ell$-block of $GL(n, q^2)$ is Morita equivalent to its Brauer correspondent by Puig’s result (see [8]).

#### 3.1. $GL(2, q^2)$ and $GU(2, q^2)$

We have isomorphisms

$$B_0(GL(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(2, q^2)), \quad B_0(GU(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(2, q^2)).$$

The blocks $B_0(SL(2, q^2))$ and $B_0(SU(2, q^2))$ have cyclic defect groups, and they are splendid equivalent by Rouquier’s result in [11]. Therefor the principal blocks $B_0(GL(2, q^2))$ and $B_0(GU(2, q^2))$ are splendid equivalent.

#### 3.2. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic $\ell > 3$.

In this case, Sylow $\ell$-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are abelian. As in case $n = 2$, we have isomorphisms

$$B_0(GL(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(3, q^2)), \quad B_0(GU(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(3, q^2)).$$

In [5], Waki and the author showed that $B_0(SU(3, q^2))$ and its Brauer correspondent, which is isomorphic to the Brauer correspondent of $B_0(SL(3, q^2))$, are splendid equivalent. Therefore $B_0(GL(3, q^2))$ and $B_0(GU(3, q^2))$ are splendid equivalent since as we mentioned above $B_0(SL(3, q^2))$ and its Brauer correspondent are Morita (Puig) equivalent by Puig’s result. Hence we also have $B_0(GL(3, q^2))$ and $B_0(GU(3, q^2))$ are splendid equivalent.

#### 3.3. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic 3.

In this case Sylow 3-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are not abelian. Our main result in this paper is the following theorem.
Theorem 3.1 (with T. Okuyama). Assume that $3^e$ divides $q + 1$ but $3^{e+1}$ does not divide $q + 1$ for $e > 0$. Then

1. The principal 3-blocks $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$ are splendid equivalent.
2. The principal 3-blocks $B_0(GL(3,q^2))$ and $B_0(GU(3,q^2))$ are splendid equivalent.
3. The principal 3-blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.
4. The principal 3-blocks $B_0(SL(3,q^2))$ are splendid equivalent.

Remark 3.1. If $e = 1$, then the result for (1) has been obtained by [6, 4, 3] and the result for (3) has been obtained by Usami and the author.

4. Outline of proof of theorem

In this section, we give an outline of a proof of Theorem 3.1 (1) and (2). Let $G = SL(3,q^2), H = SU(3,q^2), \overline{G} = PSL(3,q^2)$ and $\overline{H} = PSU(3,q^2)$. Let $P$ be a common Sylow 3-subgroup of $G$ and $H$. We denote the image of a subgroup $L$ of $G$ (or $H$) in $\overline{G}$ (or $\overline{H}$) by $\overline{L}$. For each subgroup $\overline{R}$ of $P$, let $\overline{G}_R := C_{\overline{G}}(\overline{R}), \overline{H}_R := C_{\overline{H}}(\overline{R})$, and let $\overline{M}_R$ be the Scott module of $\overline{G}_R \times \overline{H}_R$ with vertex $\Delta(\overline{R})$, where $\overline{R}$ is a Sylow 3-subgroup of $\overline{G}_R$ and $\overline{H}_R$.

(Step 1). There is essentially one subgroup of $P$(up to conjugate), which we denote by $Q$, containing $Z(P)$ such that $B_0(C_{\overline{G}}(Q))$ and $B_0(C_{\overline{H}}(Q))$ are not Morita equivalent. Then $C_{\overline{G}}(Q) \cong GL(2,q^2)$ and $C_{\overline{H}}(Q) \cong GU(2,q^2)$. Let $\overline{M}_Q \rightarrow k\overline{G}_Q \times \overline{H}_Q$ be a $\Delta(\overline{Q})$-projective cover of $k\overline{G}_Q \times \overline{H}_Q$ and $\overline{N}_Q \rightarrow \Omega_{\Delta(\overline{Q})}(k\overline{G}_Q \times \overline{H}_Q)$ be a $\Delta(\overline{Q})$-projective cover of $\Omega_{\Delta(\overline{Q})}(k\overline{G}_Q \times \overline{H}_Q)$. Then we have a splendid tilting complex for $B_0(\overline{G}_Q)$ and $B_0(\overline{H}_Q)$ of the form

$$0 \rightarrow \overline{N}_Q \rightarrow \overline{M}_Q \rightarrow 0.$$ 

For a subgroup $\overline{R}$ of $P$ not contained in $\overline{Q}$, the blocks $B_0(\overline{G}_R)$ and $B_0(\overline{H}_R)$ are Morita equivalent and the Scott module $\overline{M}_R$ gives a splendid tilting complex for these two blocks.

(Step 2). Let $M$ be the Scott module of $G \times H$ with vertex $\Delta(P)$. Let $M \rightarrow k_{G \times H}$ be a $\Delta(P)$-projective cover of $k_{G \times H}$ and $N \rightarrow \Omega_{\Delta(P)}(k_{G \times H})$ be a $\Delta(Q)$-projective cover of $\Omega_{\Delta(P)}(k_{G \times H})$. Consider the following complex

$$M^* : 0 \rightarrow N \rightarrow M \rightarrow 0.$$

and set $\overline{M}^* = \text{Inv}_{Z(P) \times 1}(M^*)$. Then the complex $\overline{M}^*$ is a splendid complex, and for each non-trivial subgroup $\overline{R}$ of $P$, the complex $\overline{M}^*(\Delta(\overline{R}))$ coincides with the complex in (Step 1). Therefore by Rouquier’s theorem (Theorem 2.1) we can see that the complex $\overline{M}^*$ induces a splendid stable equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$.

(Step 3). Let $F = - \otimes_{B_0(G)} \overline{M}^*$. The principal block of $B_0(\overline{G})$ has 5 simple modules $k, S, T_1, T_2$ and $T_3$ and the principal block of $B_0(\overline{H})$ has 5 simple modules $k, \varphi, \theta_1, \theta_2$ and $\theta_3$. Then we have the following lemma.

Lemma 4.1. There exist exact sequences

$$0 \rightarrow \Omega^{-1}(U(k, \varphi)) \rightarrow \Omega(F(S)) \rightarrow k \oplus k \rightarrow 0$$

and

$$0 \rightarrow \Omega^{-1}(U(k, \varphi, \theta_i)) \rightarrow \Omega^2(F(T_i)) \rightarrow k \rightarrow 0.$$
for \( i = 1, 2 \) and 3, where \( U(k, \varphi) \) is a uniserial module of length 2 with top \( k \), and \( U(k, \varphi, \theta_i) \) is a uniserial module of length 3 with top \( k \) and socle \( \theta_i \).

(Step 4). It follows from Lemma 4.1 that the tilting complex defined by a sequence \( \{\theta_1, \theta_2, \theta_3\}, \{\varphi, \theta_1, \theta_2, \theta_3\} \) of subsets of the set of simple modules (see [6]) gives a derived equivalence between \( B_0(G) \) and \( B_0(H) \). The equivalence is a lift of the stable equivalence given by \( F \) (see [7]), and therefore \( B_0(G) \) and \( B_0(H) \) are splendid equivalent.

Now we have the splendid tilting complex for \( B_0(G) \) and \( B_0(H) \) of the form

\[
X^\bullet: 0 \longrightarrow \overline{Q}_3 \longrightarrow \overline{Q}_2 \longrightarrow \overline{Q}_1 \oplus \overline{N} \longrightarrow \overline{M} \longrightarrow 0
\]

where \( \overline{M} = \text{Inv}_{Z(P) \times 1}(M) \) and \( \overline{N} = \text{Inv}_{Z(P) \times 1}(N) \) and \( \overline{Q}_1, \overline{Q}_2, \) and \( \overline{Q}_3 \) are projective bimodules. Since \( \text{Inv}_{Z(P) \times 1}(\cdot) \) induces a one to one correspondence between the set of trivial source \( k[G \times H] \)-modules with vertex \( \Delta(Z(P)) \) and the set of projective \( k[G \times H] \)-modules, we have a tilting complex of the form

\[
X^\bullet: 0 \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \oplus N \longrightarrow M \longrightarrow 0
\]

for \( B_0(G) \) and \( B_0(H) \), where \( Q_1, Q_2 \) and \( Q_3 \) are direct sums of trivial source \( k[G \times H] \)-modules with vertex \( \Delta(Z(P)) \) and \( \text{Inv}_{Z(P) \times 1}(X^\bullet) = X^\bullet \) (see [12, A.4]).

REFERENCES