SOME TOPICS ON DERIVED EQUIVALENT BLOCKS OF FINITE GROUPS

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1. Introduction

Let G be a finite group. Let k be an algebraically closed field of characteristic $\ell > 0$. We denote the principal block of kG by $B_0(G)$.

We say that two finite groups G and H have the same ℓ-local structure if G and H have a common Sylow ℓ-subgroup P such that whenever Q_1 and Q_2 are subgroups of P and $f : Q_1 \to Q_2$ is an isomorphism, then there is an element $g \in G$ such that $f(x) = x^g$ for all $x \in Q_1$ if and only if there is an element $h \in H$ such that $f(x) = x^h$ for all $x \in Q_1$.

There is a well known conjecture due to Broué.

Conjecture 1.1(Broué [1, 2]). Let G and H be finite groups having the same ℓ-local structure with common Sylow ℓ-subgroup P. If P is abelian then the principal blocks of G and H would be derived equivalent.

If P is not abelian, then there is a counterexample to this conjecture. However , there are some examples that P is not abelian and there is a derived equivalence between the principal blocks of G and H. We will give such examples in §3.

2. General theory

In this section, let G and H be finite groups having the same ℓ-local structure with common Sylow ℓ-subgroup P. We say that a complex of $(B_0(G), B_0(H))$-bimodules is *splendid* if each indecomposable summand of each term of the complex is a direct summand of a module of the form $kG \otimes_{kQ} kH$ for a subgroup Q of P.

Definition 2.1. Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a *splendid stable equivalence* if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{\bullet *} \cong B_0(G) \oplus Z_1, \quad X^{\bullet *} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2$$

where Z_1 and Z_2 are homotopy equivalent to complexes of projective bimodules.

Definition 2.2. Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. We say that $X^\bullet$ induces a *splendid equivalence* if we have isomorphisms

$$X^\bullet \otimes_{B_0(H)} X^{\bullet *} \cong B_0(G) \oplus Z_1, \quad X^{\bullet *} \otimes_{B_0(G)} X^\bullet \cong B_0(H) \oplus Z_2$$

where Z_1 and Z_2 are homotopy equivalent to 0. The complex $X^\bullet$ is called a *splendid tilting complex*.

The detailed version of this paper will be submitted for publication elsewhere.

By the definition, splendid equivalences induce derived equivalences and homotopy equivalences.

Theorem 2.1 (Rouquier [12]). *Let $X^\bullet$ be a splendid complex of $(B_0(G), B_0(H))$-bimodules. Then the following are equivalent.*

(1) *The complex $X^\bullet$ induces a splendid stable equivalence between $B_0(G)$ and $B_0(H)$.*

(2) *For every non-trivial subgroup Q of P, the complex $X^\bullet(\Delta(Q))$ induces a splendid equivalence between $B_0(C_G(Q))$ and $B_0(C_H(Q))$, where $\Delta(Q)$ is a diagnal subgroup and*

$$X(Q) = X^{\Delta(Q)} / \sum_{R<Q} \mathrm{Tr}_R^Q X^{\Delta(R)}.$$

In our example in §3 we will use the following method when we prove splendid equivalences.

(Step 1) Construct a splendid tilting complex between $B_0(C_G(Q))$ and $B_0(C_H(Q))$ for every non-trivial subgroup Q of P.

(Step 2) Construct a splendid stable equivalence F from $B_0(G)$ to $B_0(H)$ by gluing the splendid tilting complexes obtained in Step 1 (by using the above theorem).

(Step 3) Calculate $F(S)$ for the simple $B_0(G)$-modules.

(Step 4) Lift the stable equivalence in Step 2 to a splendid equivalence by looking at the modules calcutated in Step 3.

3. General linear groups and unitary groups

Let q be a power of a prime. Assume that ℓ is odd and ℓ^e divides $q+1$ but ℓ^{e+1} does not divide $q+1$ for some $e > 0$. Under this condition, we consider representations of the general linear group $GL(n, q^2)$ and the unitary group $GU(n, q^2)$ for small n. Note that if $\ell > n$ then the principal ℓ-block of $GL(n, q^2)$ is Morita equivalent to its Brauer correspondent by Puig's result(see [8]).

3.1. $GL(2, q^2)$ and $GU(2, q^2)$.

We have isomorphisms

$$B_0(GL(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(2, q^2)), \quad B_0(GU(2, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(2, q^2)).$$

The blocks $B_0(SL(2, q^2))$ and $B_0(SU(2, q^2))$ have cyclic defect groups, and they are splendid equivalent by Rouquier's result in [11]. Therefor the principal blocks $B_0(GL(2, q^2))$ and $B_0(GU(2, q^2))$ are splendid equivalent.

3.2. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic $\ell > 3$.

In this case, Sylow ℓ-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are abelian. As in case $n = 2$, we have isomorphisms

$$B_0(GL(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SL(3, q^2)), \quad B_0(GU(3, q^2)) \cong kZ_{\ell^e} \otimes B_0(SU(3, q^2)).$$

In [5], Waki and the author showed that $B_0(SU(3, q^2))$ and its Brauer correspondent, which is isomorphic to the Brauer correspondent of $B_0(SL(3, q^2))$, are splendid equivalent. Therefore $B_0(SL(3, q^2))$ and $B_0(SU(3, q^2))$ are splendid equivalent since as we mentioned above $B_0(SL(3, q^2))$ and its Brauer correspondent are Morita(Puig) equivalent by Puig's result. Hence we also have $B_0(GL(3, q^2))$ and $B_0(GU(3, q^2))$ are splendid equivalent.

3.3. $GL(3, q^2)$ and $GU(3, q^2)$ in characteristic 3.

In this case Sylow 3-subgroups of $GL(3, q^2)$ and $GU(3, q^2)$ are not abelian. Our main result in this paper is the following theorem.

Theorem 3.1 (with T. Okuyama). *Assume that 3^e divides $q+1$ but 3^{e+1} does not divide $q+1$ for $e > 0$. Then*
(1) *The principal 3-blocks $B_0(PSL(3,q^2))$ and $B_0(PSU(3,q^2))$ are splendid equivalent.*
(2) *The principal 3-blocks $B_0(SL(3,q^2))$ and $B_0(SU(3,q^2))$ are splendid equivalent.*
(3) *The principal 3-blocks $B_0(PGL(3,q^2))$ and $B_0(PGU(3,q^2))$ are splendid equivalent.*
(4) *The principal 3-blocks $B_0(GL(3,q^2))$ and $B_0(GU(3,q^2))$ are splendid equivalent.*

Remark 3.1. If $e = 1$, then the result for (1) has been obtained by [6, 4, 3] and the result for (3) has been obtained by Usami and the author.

4. Outline of proof of theorem

In this section, we give an outline of a proof of Theorem 3.1 (1) and (2). Let $G = SL(3,q^2)$, $H = SU(3,q^2)$, $\overline{G} = PSL(3,q^2)$ and $\overline{H} = PSU(3,q^2)$. Let P be a common Sylow 3-subgroup of G and H. We denote the image of a subgroup L of G (or H) in $\overline{G}$ (or $\overline{H}$) by $\overline{L}$. For each subgroup $\overline{R}$ of $\overline{P}$, let $\overline{G}_R := C_{\overline{G}}(\overline{R})$, $\overline{H}_R := C_{\overline{H}}(\overline{R})$, and let $\overline{M}_R$ be the Scott module of $\overline{G}_R \times \overline{H}_R$ with vertex $\Delta(\overline{R'})$, where $\overline{R'}$ is a Sylow 3-subgroup of $\overline{G}_R$ and $\overline{H}_R$.

(Step 1). There is essentially one subgroup of P(up to conjugate), which we denote by Q, containing $Z(P)$ such that $B_0(C_G(Q))$ and $B_0(C_H(Q))$ are not Morita equivalent. Then $C_G(Q) \cong GL(2,q^2)$ and $C_H(Q) \cong GU(2,q^2)$. Let $\overline{M}_Q \to k_{\overline{G}_Q \times \overline{H}_Q}$ be a $\Delta(\overline{Q'})$-projective cover of $k_{\overline{G}_Q \times \overline{H}_Q}$ and $\overline{N}_Q \to \Omega_{\Delta(\overline{Q'})}(k_{\overline{G}_Q \times \overline{H}_Q})$ be a $\Delta(\overline{Q})$-projective cover of $\Omega_{\Delta(\overline{Q'})}(k_{\overline{G}_Q \times \overline{H}_Q})$. Then we have a splendid tilting complex for $B_0(\overline{G}_Q)$ and $B_0(\overline{H}_Q)$ of the form

$$0 \longrightarrow \overline{N}_Q \longrightarrow \overline{M}_Q \longrightarrow 0.$$

For a subgroup $\overline{R}$ of $\overline{P}$ not contained in $\overline{Q}$, the blocks $B_0(\overline{G}_R)$ and $B_0(\overline{H}_R)$ are Morita equivalent and the Scott module $\overline{M}_R$ gives a splendid tilting complex for these two blocks.

(Step 2). Let M be the Scott module of $G \times H$ with vertex $\Delta(P)$. Let $M \to k_{G\times H}$ be a $\Delta(P)$-projective cover of $k_{G\times H}$ and $N \to \Omega_{\Delta(P)}(k_{G\times H})$ be a $\Delta(Q)$-projective cover of $\Omega_{\Delta(P)}(k_{G\times H})$. Consider the following complex

$$M^\bullet : \quad 0 \longrightarrow N \longrightarrow M \longrightarrow 0.$$

and set $\overline{M}^\bullet = \mathrm{Inv}_{Z(P)\times 1}(M^\bullet)$. Then the complex $\overline{M}^\bullet$ is a splendid complex, and for each non-trivial subgroup $\overline{R}$ of $\overline{P}$, the complex $\overline{M}^\bullet(\Delta(\overline{R}))$ coincides with the complex in (Step 1). Therefore by Rouquier's theorem (Theorem 2.1) we can see that the complex $\overline{M}^\bullet$ induces a splendid stable equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$.

(Step 3). Let $F = - \otimes_{B_0(G)} \overline{M}^\bullet$. The principal block of $B_0(\overline{G})$ has 5 simple modules k, S, T_1, T_2 and T_3 and the principal block of $B_0(\overline{H})$ has 5 simple modules $k, \varphi, \theta_1, \theta_2$ and θ_3. Then we have the following lemma.

Lemma 4.1. *There exist exact sequences*

$$0 \longrightarrow \Omega^{-1}(U(k,\varphi)) \longrightarrow \Omega(F(S)) \longrightarrow k \oplus k \longrightarrow 0$$

and

$$0 \longrightarrow \Omega^{-1}(U(k,\varphi,\theta_i)) \longrightarrow \Omega^2(F(T_i)) \longrightarrow k \longrightarrow 0$$

for $i = 1, 2$ and 3, where $U(k, \varphi)$ is a uniserial module of length 2 with top k, and $U(k, \varphi, \theta_i)$ is a uniserial module of length 3 with top k and socle θ_i.

(Step 4). It follows from Lemma 4.1 that the tilting complex defined by a sequence $\{\theta_1, \theta_2, \theta_3\}$, $\{\varphi, \theta_1, \theta_2, \theta_3\}$ and $\{\varphi, \theta_1, \theta_2, \theta_3\}$ of subsets of the set of simple modules (see [6]) gives a derived equivalence between $B_0(\overline{G})$ and $B_0(\overline{H})$. The equivalence is a lift of the stable equivalence given by F (see [7]), and therefore $B_0(G)$ and $B_0(H)$ are splendid equivalent.

Now we have the splendid tilting complex for $B_0(\overline{G})$ and $B_0(\overline{H})$ of the form

$$\overline{X}^\bullet : \quad 0 \longrightarrow \overline{Q}_3 \longrightarrow \overline{Q}_2 \longrightarrow \overline{Q}_1 \oplus \overline{N} \longrightarrow \overline{M} \longrightarrow 0$$

where $\overline{M} = \mathrm{Inv}_{Z(P)\times 1}(M)$ and $\overline{N} = \mathrm{Inv}_{Z(P)\times 1}(N)$ and $\overline{Q}_1$, $\overline{Q}_2$, and $\overline{Q}_3$ are projective bimodules. Since $\mathrm{Inv}_{Z(P)\times 1}(-)$ induces a one to one correspondence between the set of trivial source $k[G \times H]$-modules with vertex $\Delta(Z(P))$ and the set of projective $k[\overline{G} \times \overline{H}]$-modules, we have a tilting complex of the form

$$X^\bullet : \quad 0 \longrightarrow Q_3 \longrightarrow Q_2 \longrightarrow Q_1 \oplus N \longrightarrow M \longrightarrow 0$$

for $B_0(G)$ and $B_0(H)$, where Q_1, Q_2 and Q_3 are direct sums of trivial source $k[G \times H]$-modules with vertex $\Delta(Z(P))$ and $\mathrm{Inv}_{Z(P)\times 1}(X^\bullet) = \overline{X}^\bullet$ (see [12, A.4]).

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