NAKAYAMA ISOMORPHISMS FOR THE MAXIMAL QUOTIENT RING OF A LEFT HARADA RING

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Abstract. From several results of Kado and Oshiro, we see that if the maximal quotient ring of a given left Harada ring R of type $(*)$ has a Nakayama automorphism, then R has a Nakayama isomorphism. This result poses a question whether if the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism, then R has a Nakayama isomorphism. In this paper, we shall show that a basic ring of the maximal quotient ring of a given Harada ring has a Nakayama isomorphism if and only if its Harada ring has a Nakayama isomorphism.

Introduction

Let R be a basic left Harada ring. Then we have a complete set

$$\{e_{11}, \ldots, e_{1n(1)}, \ldots, e_{m1}, \ldots, e_{mn(m)}\}$$

of primitive idempotents for R such that for each $i = 1, \ldots, m$

(a) $e_{i1}R$ is injective as a right R-module;
(b) $J(e_{i,k-1}R) \cong e_{ik}R$ for each $k = 2, \ldots, n(i)$.

We call R a ring of type $(*)$ if there exists an unique g_i in $\{e_{in(i)}\}_{i=1}^{m}$ for each $i = 1, \ldots, m$ such that the socle of $e_{i1}R$ is isomorphic to $g_iR/J(g_iR)$ and the socle of Rg_i is isomorphic to $Re_{i1}/J(Re_{i1})$.

Oshiro [10] showed the following;

Result A ([10, Theorem 2]). *Suppose that R is a left Harada ring which is not of type $(*)$. Then there exists a series of left Harada rings and surjective ring homomorphisms:*

$$T_1 \overset{\phi_1}{\twoheadrightarrow} T_2 \overset{\phi_2}{\twoheadrightarrow} \cdots \overset{\phi_{n-1}}{\twoheadrightarrow} T_n \overset{\phi_n}{\twoheadrightarrow} R$$

such that

(1) *T_1 is of type $(*)$, and*
(2) Ker ϕ_i *is a simple ideal of T_i for any $i \in \{1, \ldots, n\}$.*

Kado and Oshiro [7] showed the following results;

Result B ([7, Proposition 5.3]). *If every basic* QF *rings has a Nakayama automorphism, then every basic left Harada ring of type $(*)$ has a Nakayama isomorphism.*

The detailed version of this paper will be submitted for publication elsewhere.

Result C ([7, Proposition 5.4]). *Let S be a two-sided ideal of R that is simple as a left ideal and as a right ideal. If R has a Nakayama isomorphism, then R/S has a Nakayama isomorphism.*

Moreover Kado showed the following;

Result D ([6, Corollary]). *The maximal quotient ring of a left Harada ring of type $(*)$ is a QF ring.*

Using these four results, we see that if the maximal quotient ring of a given left Harada ring R of type $(*)$ has a Nakayama automorphism, then R has a Nakayama isomorphism. So this result poses a question whether if the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism, then R has a Nakayama isomorphism. In this paper, we shall show that the maximal quotient ring of a given left Harada ring R has a Nakayama isomorphism iff R has a Nakayama isomorphism.

Throughout this paper, we assume that all rings are associative rings with identity and all modules are unitary. By M_R (resp. ${}_RM$), we means that M is a right (resp. left) R-module, respectively. We denote the set of primitive idempotents of R by $\mathrm{Pi}(R)$, and denote a complete set of primitive idempotents of R by $\mathrm{pi}(R)$.

We call a one-sided artinian ring R right (resp. left) QF-3 ring if $E(R_R)$ (resp. $E({}_RR)$) is projective, respectively.

We denote the maximal left (resp. right) quotient ring of R by $Q_\ell(R)$ (resp. $Q_r(R)$), respectively, and denote the maximal left and maximal right quotient ring of R by $Q(R)$. If a ring is QF-3, its maximal left quotient ring and its right quotient ring coincide by [16, Theorem 1.4].

1. Maximal quotient ring

We list some basic results, which several authors showed, for our main result in this paper. Recall that for $e, f \in \mathrm{Pi}(R)$, we say that the pair $(eR : Rf)$ is an i-pair if $\mathrm{S}(eR) \cong fR/J(fR)$ and $\mathrm{S}(Rf) \cong Re/J(Re)$.

Lemma 1 ([5]). *Let R be a one-sided artinian ring, and let $e \in \mathrm{Pi}(R)$. Then the following conditions are equivalent:*

(1) *eR is injective.*

(2) *There exists some $f \in \mathrm{Pi}(R)$ such that $(eR : Rf)$ is an i-pair.*

In this case, Rf is also injective.

Let R be a left perfect ring. Then R has a primitive idempotent e with $S(R_R)e \neq 0$. If R is QF-3, then the primitive idempotent e with $S(R_R)e \neq 0$ are characterized as follows;

Lemma 2 ([4, Theorem 2.1]). *Let R be a one-sided artinian* QF-3 *ring, and let $e \in \mathrm{Pi}(R)$. Then ${}_RRe$ is injective if and only if $\mathrm{S}(R_R)e \neq 0$.*

We call $e \in \mathrm{Pi}(R)$ right (resp. left) S-primitive if $\mathrm{S}(R_R)e \neq 0$ (resp. $e\,\mathrm{S}({}_RR) \neq 0$), respectively.

The following statement, which Storrer [15, Proposition 4.8] showed, is helpful in this paper.

Lemma 3 ([15, Proposition 4.8]). *Let R and $Q = Q(R)$ be left perfect. Then*

(1) *If e is a right S-primitive idempotent for R, then so is it for Q.*

(2) *if* e_1, e_2 *are right* S*-primitive idempotents for* R*, then* $e_1R \cong e_2R$ *if and only if* $e_1Q \cong e_2Q$*.*
(3) *If* e *is a right* S*-primitive idempotent for* Q*, then there exists a right* S*-primitive idempotent* $e' \in R$ *such that* $eQ \cong e'Q$*.*

A ring R is called a left Harada ring if it is left artinian and its complete set $\mathrm{pi}(R)$ of orthogonal primitive idempotents is arranged as follows:

$$\mathrm{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)},$$

where

(a) each $e_{i1}R_R$ is an injective module for each $i = 1, 2, \ldots, m$.
(b) $e_{i,k-1}R_R \cong e_{ik}R$, or $J(e_{i,k-1}R_R) \cong e_{ik}R$ for each i and each $k = 2, 3, \ldots, n(i)$.
(c) $e_{ik}R \not\cong e_{jt}R$ for $i \neq j$.

Remark 1. Let R be a left Harada ring. Then $Q(R)$ is also a left Harada ring (See [6, Theorem 4]) and a complete set $\mathrm{pi}(Q)$ of orthogonal primitive idempotents for Q coincides with $\mathrm{pi}(R)$ (See [6, p.248]).

Using Remark 1, Kado showed the following;

Proposition 4 ([6, Proposition 2])**.** *Let* R *be a left Harada ring, and let* $(eR : Rf)$ *be an i-pair for* $e, f \in \mathrm{pi}(R)$*. Then* $(eQ(R) : Q(R)f)$ *is an i-pair .*

Recall the following notation [6, p.249]. Let $\theta : fR \to eR$ be an R-monomorphism such that $\mathrm{Im}\,\theta = J(eR)$, where $e, f \in \mathrm{Pi}(R)$. Then by [15, Proposition 4.3], θ can be uniquely extended to a $Q_r(R)$-homomorphism $\theta^* : fQ_r(R) \to eQ_r(R)$.

We shall need the following results.

Lemma 5 ([6, Proposition 3])**.** *Let* R *be a basic and left Harada ring, and* $Q = Q(R)$ *and* θ *as above. Then the following hold.*

(1) *If* e *is not right* S*-primitive, then the extension* $\theta^* : fQ \to eQ$ *is an isomorphism.*
(2) *If* e *is right* S*-primitive, then the extension* $\theta^* : fQ \to eQ$ *is a monomorphism such that* $\mathrm{Im}\,\theta^* = J(eQ)$*.*

Remark 2 (cf. [15, Lemma 4.2])**.** Let $\{g_i\} \cup \{f_j\}$ be a complete set of orthogonal primitive idempotents for R, where the g_i are right S-primitive and the f_j are not right S-primitive. We denote g_0 by $g_0 = \sum g_i$. Then $Q(R)g_0 = Rg_0$ and $Q(R)g = Rg$ for every right S-primitive idempotent g of R.

Let R be a basic left artinian ring, and let $\{e_1, e_2, \ldots , e_n\}$ be a complete set of orthogonal primitive idempotents for R and let

$$S = \mathrm{End}_R(\oplus_{i=1}^{n} E(Re_i/J(Re_i)))$$

be the endomorphism ring of a minimal injective cogenerator for R-mod. Let f_i be the primitive idempotent for S corresponding to the projection

$$\oplus_{i=1}^{n} E(Re_i/J(Re_i)) \to E(Re_i/J(Re_i)).$$

Then we call a ring isomorphism $\tau : R \to S$ a Nakayama isomorphism if $\tau(e_i) = f_i$ for each $i = 1, 2, \ldots, n$. By [3, p.42], the existence of a Nakayama isomorphism does not depend on the choice of the complete set $\{e_1, e_2, \ldots, e_n\}$ of orthogonal primitive idempotents. (See [7, Remark on p.387].)

It is important whether the maximal quotient ring of a basic artinian ring is basic since a Nakayama isomorphism is defined on a basic ring. Here we shall study the case that the maximal quotient ring of a given left Harada ring is basic.

Theorem 6 (cf. [2, Corollary 22])**.** *Let R be a basic and left Harada ring and $Q = Q(R)$. Then Q is a basic ring if and only if R either is* QF *or satisfies the following; $n(i) = 1$ or 2 and Re_{i1} is injective for any i. In this case $R = Q$.*

Proof. Note that both R and Q are artinian QF-3. Let $\mathrm{pi}(R) = \bigcup_{i=1}^{m}\{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotent for R satisfying the following conditions:

(a) $e_{i1}R_R$ is injective for each $i = 1, 2, \ldots, m$,
(b) $e_{i,j+1}R_R \cong J(e_iR_R)$ for $j = 1, 2, \ldots, n(i) - 1$.

We have a complete set $\{Rg_1, \ldots, Rg_m\}$ of pairwise non-isomorphic indecomposable injective projective left R-modules, such that the $(e_{i1}R : Rg_i)$ are i-pair for each $i = 1, \ldots, m$ since R is basic and artinian QF-3.

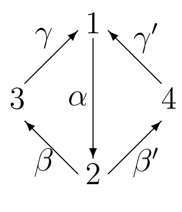
Assume that Q is basic. Let $e_{i,k+1}, e_{ik} \in \{e_{ij}\}_{j=2}^{n(i)}$. Then we have an R-monomorphism $\theta_{ik} : e_{i,k+1}R \to e_{ik}R$ such that $\mathrm{Im}\,\theta = J(e_{ik}R)$. If e_{ik} is not right S-primitive, then $e_{ik+1}Q \cong e_{ik}Q$ by Lemma 5. This contradicts that Q is basic. Hence e_{ik} is right S-primitive for $k = 1, 2, \ldots, n(i) - 1$. Since the Re_{ik} are injective for each $k = 1, 2, \ldots, n(i) - 1$ by Lemma 2, there exists some Rg in $\{Rg_1, \ldots, Rg_m\}$ such that $Re_{ik} \cong Rg$. However R is basic, so we see that $n(i) \leq 2$ and e_{i1} is right S-primitive.

In case $n(i) = 1$ for every $i = 1, \ldots, m$, then R is QF.

In case $n(i) = 2$ for some $i \in \{1, \ldots, m\}$. If $e_{in(i)}$ is right S-primitive, then ${}_RRe_{in(i)}$ is injective by Lemma 2. Hence $e_{in(i)}$ is not right S-primitive since ${}_RRe_{i1}$ is injective and so $\{Rg_1, \ldots, Rg_m\} = \{Re_{11}, \ldots, Re_{m1}\}$.

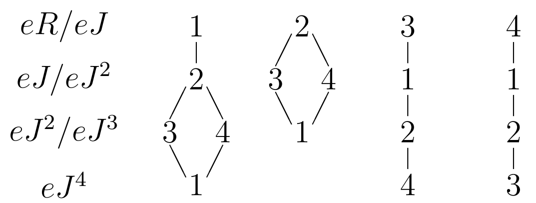
Conversely, first, assume that R is QF. Since ${}_RRe$ is injective for any $e \in \mathrm{pi}(R)$, e is right S-primitive by Lemma 2. Thus, $eQ \ncong fQ$ for any $e, f \in \mathrm{pi}(R) = \mathrm{pi}(Q)$ by Lemma 3. Therefore Q is basic. Next, assume that R satisfies $n(i) = 1$ or 2 and Re_{i1} is injective for any i. Then e_{i1} is left S-primitive and so $eQ = eR$ by Remark 2. Hence $J(eQ) = J(eR)$. Therefore it is also clear to see that $R = Q$. $\square$

Example 1. We shall give a basic left Harada ring R with $J(R)^5 = 0$, which is not QF. Let R be an algebra over a field K defined by the following quiver;



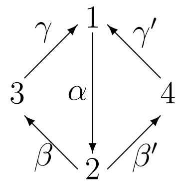
with the relations $\gamma\beta = \gamma'\beta'$, $\alpha\gamma\beta = 0$, and $\beta'\alpha\gamma = 0$.

The composition diagrams of the Loewy factors of the indecomposable projective modules of R_R is the following.

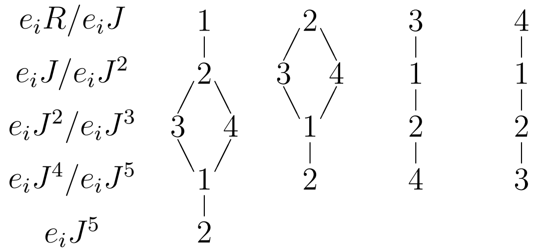


Then R is a left Harada ring which is not QF since e_1R_R, e_3R_R and e_4R_R are injective and $e_2R_R \cong J(e_1R)$. Moreover e_1, e_3, e_4 are right S-primitive. Hence $e_1Q(R) = e_1R$, $e_3Q(R) = e_3R$ and $e_4Q(R) = e_4R$ are injective and $e_2Q(R) \cong J(e_1Q(R))$. Therefore $R = Q(R)$.

Example 2. We shall give a basic Harada ring R with $J(R)^6 = 0$, but $Q(R)$ is not basic. Let R be an algebra over a field K defined by the following quiver;



with the relations $0 = \beta\alpha\gamma\beta = \beta'\alpha\gamma'\beta' = \beta\alpha\gamma = \beta'\alpha\gamma'$, and $\gamma\beta = \gamma'\beta'$. Then the composition diagrams of the Loewy factors of the indecomposable projective modules of R_R is the following.



Then since e_1R_R, e_3R_R and e_4R_R are injective and $e_2R_R \cong J(e_1R)$, R is a left Harada ring which is not QF. Hence $e_2Q(R) \cong e_1Q(R)$ since e_1 is not right S-primitive. Therefore $Q(R)$ is not basic.

2. Nakayama isomorphism

In this section, we study the Nakayama isomorphisms for the representative matrix ring of a basic left Harada ring and its maximal quotient ring. Let R be a basic left Harada ring, and let $\mathrm{pi}(R) = \bigcup_{i=1}^{m}\{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of orthogonal primitive idempotents as in Theorem 6. Furthermore, let R^* be the representative matrix ring of R. R^* is represented as block matrices as follows:

$$R^* = \begin{pmatrix} R^*_{11} & \cdots & R^*_{1m} \\ & \cdots & \\ R^*_{m1} & \cdots & R^*_{mm} \end{pmatrix},$$

where $R^*_{ij} = P_{ij}$ for $j \neq \sigma(i)$ and $R^*_{i\sigma(i)} = P^*_{i\sigma(i)}$ (See [7, Section 4]).

Here, adding one row and one column to R^*, we make an extended matrix ring $W_i(R)$ of R as follows:

$$\begin{pmatrix} R^*_{11} & \cdots & \cdots & R^*_{1i} & Y_1 & R^*_{1,i+1} & \cdots & R^*_{1m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{i1} & \cdots & \cdots & R^*_{ii} & Y_i & R^*_{i,i+1} & \cdots & R^*_{im} \\ X_1 & \cdots & X_{i-1} & X_i & Q & X_{i+1} & \cdots & X_m \\ R^*_{i+1,1} & \cdots & \cdots & R^*_{i+1,i} & Y_{i+1} & R^*_{i+1,i+1} & \cdots & R^*_{i+1,m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{m1} & \cdots & \cdots & R^*_{mi} & Y_m & R^*_{m,i+1} & \cdots & R^*_{mm} \end{pmatrix},$$

where X_k is the last row of R^*_{ik} ($k = 1, \ldots, m,\ k \neq i$), Y_k is the last column of R^*_{ki} ($k = 1, \ldots, m$), $X_i = (P^*_{in(i),i1} \cdots P^*_{in(i),in(i)-1} J(P^*_{in(i),in(i)}))$, and $Q = P^*_{in(i),in(i)}$.

Then $W_i(R)$ naturally becomes a ring by operations of R^*. We call this the *i-th extended ring* of R.

Proposition 7 ([7, Proposition 5.11]). *If $W_i(R)$ has a Nakayama isomorphism, then R also has a Nakayama isomorphism.*

Let R be a basic and left Harada ring, and let

$$\mathrm{pi}(R) = \bigcup_{i=1}^{m} \{e_{ij}\}_{j=1}^{n(i)}$$

be a complete set of orthogonal primitive idempotents of R satisfying the following;

(1) $e_{i1}R_R$ is injective for each $i = 1, 2, \ldots, m$.

(2) $e_{ij}R \cong J(e_{i,j-1}R)$ for each $j = 2, \ldots, n(i)$.

Then (See [7, p.388]), for any e_{ij} in $\mathrm{pi}(R)$, there exists some g_i in $\mathrm{pi}(R)$ with Rg_i injective such that $E(Re_{ij}/J(Re_{ij})) \cong Rg_i/S_{j-1}(Rg_i)$, where $S_j(Rg_i)$ is the j-th socle of Rg_i. We denote the generator $g_i + S_{j-1}(Rg_i)$ of $Rg_i/S_{j-1}(Rg_i)$ by g_{ij} for each $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n(i)$. Then by [7, Proposition 3.2], a minimal injective cogenerator $G = \oplus_{i,j} Rg_{ij}$ is finitely generated. Therefore we note that R is left Morita dual to $\mathrm{End}_R(G)$ by [1, Theorem 30.4]. We call this $\mathrm{End}(_RG)$ the *dual ring* of R. We denote the dual ring of R by $T(R)$.

For the proof of proposition 8 below, we denote

$$\begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & \cdots & 0 & R^*_{ij} & 0 & \cdots & 0 \\ 0 & & & \cdots & & & 0 \end{pmatrix} \subseteq R^*$$

by $[R^*_{ij}]$ and

$$\begin{pmatrix} 0 & & & \cdots & & & 0 \\ 0 & \cdots & 0 & R^*_{ij} & 0 & \cdots & 0 \\ 0 & & & \cdots & & & 0 \end{pmatrix} \subseteq W_i(R)$$

by $[R^*_{ij}]^w$.

By using the result that Kado and Oshiro [7, Proposition 5,11] showed, we shall show the following proposition. The proposition is essential in this paper.

Proposition 8. *$W_i(R)$ has a Nakayama isomorphism if and only if so does R.*

Proof. ($\Rightarrow$). By Proposition 7 ([7, Proposition 5,11]). ($\Leftarrow$). As [7, Proposition 5.11], let e_{ij} be the matrix of R^* such that the (ij, ij)-component is the unity and other components are zero, and let w_{ij} be the matrix of $W_i(R)$ such that the (ij, ij)-component is the unity and other components are zero. Note that the size of the columns in $W_i(R)$ is $n(i) + 1$. Let Ψ be the natural embedding homomorphism;

$$\begin{pmatrix} R^*_{11} & \cdots & R^*_{1m} \\ & \cdots & \\ R^*_{m1} & \cdots & R^*_{mm} \end{pmatrix}$$

$$\downarrow \Psi$$

$$\begin{pmatrix} R^*_{11} & \cdots & \ldots & R^*_{1i} & 0 & R^*_{1,i+1} & \cdots & R^*_{1m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{i1} & \cdots & \cdots & R^*_{ii} & 0 & R^*_{i,i+1} & \cdots & R^*_{im} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ R^*_{i+1,1} & \cdots & \cdots & R^*_{i+1,i} & 0 & R^*_{i+1,i+1} & \cdots & R^*_{i+1,m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ R^*_{m1} & \cdots & \cdots & \underbrace{R^*_{mi} & 0}_{i+1} & R^*_{m,i+1} & \cdots & R^*_{mm} \end{pmatrix},$$

where $R^*_{ij} \to R^*_{ij}$ are identity maps for all i, j. Moreover let h_{ij} be the matrix of $T(R)$ such that the (ij, ij)-component is the unity and other components are zero, and let v_{ij} be the matrix of $W_i(T(R))$ such that the (ij, ij)-component is the unity and other components are zero. Note that the size of the columns in $W_i(T(R))$ is $n(i) + 1$. Let

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1m} \\ & \cdots & \\ T(R)_{m1} & \cdots & T(R)_{mm} \end{pmatrix}$$

be the representative matrix ring $T(R)^*$ of $T(R)$, and let $T(W_i(R))$ be the dual ring of $W_i(R)$ as follows;

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1i} & {}^tY_1 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{i1} & \cdots & T(R)_{ii} & {}^tY_i & T(R)_{i,i+1} & \cdots & T(R)_{im} \\ {}^tX_1 & \cdots & {}^tX_i & {}^tQ & {}^tX_{i+1} & \cdots & {}^tX_m \\ T(R)_{i+1,1} & \cdots & T(R)_{i+1,i} & {}^tY_{i+1} & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{m1} & \cdots & T(R)_{mi} & {}^tY_m & T(R)_{m,i+1} & \cdots & T(R)_{mm} \end{pmatrix}.$$

Letting $\Psi_{T(R)}$ be the natural embedding homomorphism;

$$\begin{pmatrix} T(R)_{11} & \cdots & T(R)_{1m} \\ & \cdots & \\ T(R)_{m1} & \cdots & T(R)_{mm} \end{pmatrix}$$

$$\downarrow \Psi_{T(R)}$$

$$\begin{pmatrix} T(R)_{11} & \cdots & \ldots & T(R)_{1i} & 0 & T(R)_{1,i+1} & \cdots & T(R)_{1m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{i1} & \cdots & \cdots & T(R)_{ii} & 0 & T(R)_{i,i+1} & \cdots & T(R)_{im} \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ T(R)_{i+1,1} & \cdots & \cdots & T(R)_{i+1,i} & 0 & T(R)_{i+1,i+1} & \cdots & T(R)_{i+1,m} \\ \vdots & & & \vdots & \vdots & \vdots & & \vdots \\ T(R)_{m1} & \cdots & \cdots & \underbrace{T(R)_{mi} & 0}_{i+1} & T(R)_{m,i+1} & \cdots & T(R)_{mm} \end{pmatrix},$$

where $T(R)_{ij} \to T(R)_{ij}$ are identity maps for all i, j. We note that $T(W_i(R)) = W_i(T(R))$ (See [7, Proposition 5.11]).

Assume that $\varphi : R^* \to T(R)^*$ is a Nakayama isomorphism with $\varphi(e_{ij}) = h_{ij}$. (i.e., $\varphi([r_{kl}]) \in [T(R)_{kl}]$ for any $[r_{kl}] \in [R^*_{ij}]$, where (k,l)-componentwise of R^*_{ij} corresponds to (k,l)-componentwise of $T(R)_{ij}$.) We consider the following diagram;

$$\begin{array}{ccc} W_i(R) & & W_i(T(R)) \\ \Psi \uparrow & & \uparrow \Psi_{T(R)} \\ R^* & \xrightarrow{\varphi} & T(R). \end{array}$$

Here we define a map $\bar{\varphi} : W_i(R) \to W_i(T(R))$ as follows;

(a) $\bar{\varphi}([r_{kl}]^w) = [\varphi([r_{kl}])]^w \in [T(R)_{kl}]^w$
for any $[r_{kl}]^w \in [R^*_{kl}]^w; 1 \le k \le m, 1 \le l \le m$;
(b) $\bar{\varphi}([x]^w) \in [{}^tX_k]^w$ for any $[x]^w \in [X_k]; k = 1, \dots, m$;
(c) $\bar{\varphi}([y]^w) \in [{}^tY_l]^w$ for any $[y]^w \in [Y_l]^w; l = 1, \dots, m$;
(d) $\bar{\varphi}([q]^w) \in [{}^tQ]^w$ for any $[q]^w \in [Q]^w$.

Since $\varphi(e_{ij}) = h_{ij}$, $\bar{\varphi}$ is well-defined. Moreover it is satisfied $\bar{\varphi}(w_{i,n(i)+1}) = f_{i,n(i)+1}$. Then we can easily check that $\bar{\varphi}$ is a Nakayama isomorphism. □

Remark 3. We shall define a special case of an extended ring for a given ring R. Let $\{e_1, e_2, \dots, e_n\}$ be a complete set of orthogonal primitive idempotents for R. Then for primitive idempotent e_i in R, we define R_{e_i} as follows;

$$\begin{pmatrix} e_1Re_1 & \cdots & e_1Re_i & Y_1 & e_1Re_{i+1} & \cdots & e_1Re_n \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \\ e_iRe_1 & \cdots & e_iRe_i & Y_i & e_iRe_{i+1} & \cdots & e_iRe_n \\ X_1 & \cdots & X_i & U & X_{i+1} & \cdots & X_n \\ e_{i+1}Re_1 & \cdots & e_{i+1}Re_i & Y_{i+1} & e_{i+1}Re_{i+1} & \cdots & e_{i+1}Re_n \\ \vdots & & \vdots & \vdots & \vdots & \ddots & \\ e_nRe_1 & \cdots & e_nRe_i & Y_n & e_nRe_{i+1} & \cdots & e_nRe_n \end{pmatrix},$$

where the X_j are e_iRe_j for $j = 1, \dots, i-1, i+1, \dots, n$, X_i is $J(e_iRe_i)$, the Y_k are e_kRe_i for $k = 1, \dots, n$ and U is e_iRe_i. Then R_{e_i} is a ring by usual matrix operations.

Remark 4. Proposition 8 says that a basic left Harada ring R has a Nakayama isomorphism if and only if so does R_e for $e \in \mathrm{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$.

We denote a basic ring of $Q(R)$ by $\mathrm{Q}^b(R)$.

Remark 5. If R is a one-sided artinian QF-3 ring, the number of right S-primitive idempotents for R coincides with that of left S-primitive idempotents for R.

Theorem 9. *Let R be a basic and left Harada ring and let $Q = Q(R)$. Then Q has a Nakayama isomorphism if and only if so does R.*

Proof. If Q is basic, then $R = Q$ by Theorem 6. Hence we may assume that Q is not basic. Let $\mathrm{pi}(R) = \bigcup_{i=1}^m \{e_{ij}\}_{j=1}^{n(i)}$ be a complete set of primitive idempotents for R as given in the proof of Theorem 6. Then if $\{e_{ij}\}_{j=1}^{n(i)}$ has no right S-primitive idempotents, then $e_{i1}Q \cong e_{ij}Q$ for $j = 2, \dots, n(i)$ by Lemma 5. If $\{e_{ij}\}_{j=1}^{n(i)}$ has only one right S-primitive idempotent, say e_{ik}, then

$$\begin{cases} e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \dots, k; \\ e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\ e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k+2, \dots, n(i). \end{cases}$$

Moreover, if $\{e_{ij}\}_{j=1}^{n(i)}$ has two right S-primitive idempotents, say, e_{ik}, e_{it} $(k < t)$, then

$$\begin{cases} e_{i1}Q \cong e_{ij}Q & \text{for } j = 2, \dots, k; \\ e_{i,k+1}Q \cong J(e_{ik}Q) & \text{and} \\ e_{i,k+1}Q \cong e_{ij}Q & \text{for } j = k+2, \dots, t; \\ e_{i,t+1}Q \cong J(e_{it}Q) & \text{and} \\ e_{i,t+1}Q \cong e_{ij}Q & \text{for } j = t+2, \dots, n(i). \end{cases}$$

Repeating the same argument and Remark 5, we have the following sequences for $i = 1, \dots, m$;

$$\begin{array}{ccccccc} e_{i1}Q & > & e_{i1}J(Q) & & & & \\ & & \wr\uparrow & & & & \\ & & e_{i,k_1+1}Q & > & J(e_{i,k_1+1}Q) & & \\ & & & & \wr\uparrow & & \\ & & & & e_{i,k_2+1}Q & & \cdots, \end{array}$$

where e_{ik_t} is right S-primitive. Hence the complete set of the primitive idempotents $\mathrm{pi}(Q^b)$ for Q^b is $\bigcup_{i=1}^{m}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \subseteq \mathrm{pi}(R) = \mathrm{pi}(Q)$ and $e_{i1}\, Q^b$ is injective. Since e_{i1} is left S-primitive, $e_{i1}R = e_{i1}Q$ by Remark 2 and so $e_{i1}Re_{i1} = e_{i1}Qe_{i1}$. Hence we have a ring isomorphism from Q^b to a subring of R.

(i) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ with $e_{hn(h)}$ right S-primitive. We put $e_h = e_{h1} + \cdots + e_{hn(h)}$. Then by Lemma 3 and Lemma 5, we $e_hR = e_hQ$. (ii) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ without right S-primitive. By Remark 3, $Q^b_{e_{h1}}$ is isomorphism to a ring with the complete set $\bigcup_{i\neq h}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \cup \{e_{h1}, e_{h2}\}$ of primitive idempotents. Similarly repeating $n(h)-2$ times, we can make an extended ring with the complete set $\cup_{i\neq h}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1} \cup \{e_{hj}\}_{j=1}^{n(h)}$ of primitive idempotents.

Letting,

$$
\mathrm{Q}^b = \begin{pmatrix}
* & & e_{11}Re_{h1} & & * & \\
& & \vdots & & & \\
e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\
& & \vdots & & & \\
* & & e_{m1}Re_{h1} & & * & \\
& & \vdots & & &
\end{pmatrix},
$$

$$
\mathrm{Q}^b_{e_{h1}} = \begin{pmatrix}
* & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * & \\
& & \vdots & \vdots & & & \\
e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\
e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\
& & \vdots & \vdots & & & \\
* & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * & \\
& & \vdots & \vdots & & &
\end{pmatrix}.
$$

For two submodules

$$
A = \begin{array}{r} \\ h1 > \\ \\ \end{array} \begin{pmatrix}
0 & & \cdots & \cdots & & & 0 \\
e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\
0 & & \cdots & \cdots & & & 0
\end{pmatrix},
$$

$$
B = \begin{array}{r} \\ h1 > \\ \\ \\ \end{array} \begin{pmatrix}
0 & & \cdots & \cdots & & & 0 \\
0 & & \cdots & \cdots & & & 0 \\
e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\
0 & & \cdots & \cdots & & & 0
\end{pmatrix}
$$

of $\mathrm{Q}^b_{e_{h1}}$, $J(A) \cong B$ by [13, Theorem 1].

Hence as a ring isomorphism,

$$\begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h2} & & * & \\ & & \vdots & \vdots & & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h2} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h2}Re_{11} & \dots & e_{h2}Re_{h1} & e_{h2}Re_{h2} & \dots & e_{h2}Re_{m1} & \dots \\ & & \vdots & \vdots & & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h2} & & * & \\ & & \vdots & \vdots & & & \end{pmatrix}$$

$$\cong$$

$$\begin{pmatrix} * & & e_{11}Re_{h1} & e_{11}Re_{h1} & & * & \\ & & \vdots & \vdots & & & \\ e_{h1}Re_{11} & \dots & e_{h1}Re_{h1} & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ e_{h1}Re_{11} & \dots & J(e_{h1}Re_{h1}) & e_{h1}Re_{h1} & \dots & e_{h1}Re_{m1} & \dots \\ & & \vdots & \vdots & & & \\ * & & e_{m1}Re_{h1} & e_{m1}Re_{h1} & & * & \\ & & \vdots & \vdots & & & \end{pmatrix}$$

by [13, Theorem 1] again.

(iii) We choose $\{e_{h1}\}_{h=1}^{n(h)} \subset \mathrm{pi}(R)$ with some right S-primitive idempotents. Then we denote a right S-primitive idempotent of $\{e_{h1}\}_{h=1}^{n(h)}$ by e_{hk_t}. We reset

$$\{e_{h1}\}_{h=1}^{n(h)} = \{e_{h1}, \dots, e_{hk_1}, \dots, e_{hk_2}, \dots\}.$$

Then the complete set $\mathrm{pi}(\mathrm{Q}^b)$ of Q^b is $\bigcup_{i=1}^{m}\{e_{i1}, e_{i,k_t+1}\}_{t\geq 1}$. First by the same argument above for e_{i1}, e_{i,k_1+1}, we have a ring isomorphic to a ring with the complete set $\{e_{i1}, \dots, e_{i,k_1+1}\} \subset \mathrm{pi}(R)$. Next, by [13, Theorem 1], repeating the same argument like (ii), for e_{i,k_1+1}, e_{i,k_2+1}, we have a ring isomorphism to a ring with the complete set $\{e_{i1}, \dots, e_{ik_1}, e_{ik_1+1}, \dots, e_{ik_2}, e_{i,k_2+1}\}$. Hence the suitable extended ring of Q^b is isomorphic to R. Therefore, by Proposition 8, Q^b has a Nakayama isomorphism if and only if so does R. □

3. Another question

Oshiro's result(Result A) in the introduction also poses another question whether there exist surjective ring homomorphisms:

$$\begin{array}{ccccccccc} Q(T_1) & \overset{\bar\phi_1}{\twoheadrightarrow} & Q(T_2) & \overset{\bar\phi_2}{\twoheadrightarrow} & \dots & \overset{\bar\phi_{n-1}}{\twoheadrightarrow} & Q(T_n) & \overset{\bar\phi_n}{\twoheadrightarrow} & Q(R) \\ \vee & & \vee & & & & \vee & & \vee \\ T_1 & \overset{\phi_1}{\twoheadrightarrow} & T_2 & \overset{\phi_2}{\twoheadrightarrow} & \dots & \overset{\phi_{n-1}}{\twoheadrightarrow} & T_n & \overset{\phi_n}{\twoheadrightarrow} & R. \end{array}$$

However K. Koike informed the author the following examples;

Example 3. Let Q be a local serial ring, and $J(Q) \neq 0, J(Q)^2 = 0$. Then $J(Q) = S(Q)$. We put

$$R = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & J \end{pmatrix},$$

where $J = J(Q)$. Then R is a serial ring of an admissible sequence (3,2) and so we see that $R = Q(R)$. Also

$$T_1 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix}, \quad T_2 = \begin{pmatrix} Q & Q \\ J & Q \end{pmatrix} / \begin{pmatrix} 0 & J \\ 0 & 0 \end{pmatrix},$$
$$Q(T_1) = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}, \qquad Q(T_2) = T_2.$$

$\begin{pmatrix} J & J \\ J & J \end{pmatrix}$ is a unique non-trivial ideal of $Q(T_1)$. Hence there does not exist a surjective ring homomorphism $Q(T_1)$ to $Q(T_2)$.

Example 4. We put

$$T = \begin{pmatrix} \mathbf{K} & \mathbf{K} & \mathbf{K} \\ 0 & \mathbf{K} & \mathbf{K} \\ 0 & 0 & \mathbf{K} \end{pmatrix}, I = \begin{pmatrix} 0 & 0 & \mathbf{K} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\mathbf{K}$ is a field, and $R = T/I$. Then R is a serial ring of an admissible sequence (2,2,1) and we have a natural map

$$T = T_1 \to R.$$

However the maximal quotient ring $Q(T)$ of T is the full matrix algebra with degree 3 over a field $\mathbf{K}$ and $Q(R) = R$. Since $Q(T)$ is semisimple, there does not exist a surjective ring homomorphism $Q(T)$ to $Q(R)$.

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