# THE HOLONOMIC RANK FORMULA FOR $A$-HYPERGEOMETRIC SYSTEM [1]

Go Okuyama

## 1. Introduction

Given a finite set $A$ of $d$-dimensional integral vectors which belong to one hyperplane off the origin in $\mathbb{Q}A$ and a parameter vector $\beta \in \mathbb{C}^d$, Gel'fand, Kapranov and Zelevinskii [5] defined a system of differential equations, called an $A$-hypergeometric system $M_A(\beta)$. They proved that the holonomic rank of $M_A(\beta)$ equals the normalized volume of the convex hull of $A$ and the origin (denote by $\mathrm{vol}(A)$) for any $\beta$ when the semigroup ring $\mathbb{C}[\mathbb{N}A]$ determined by $A$ is Cohen-Macaulay. In general, the rank is not less than the volume (see [1], [13], Theorem 3.5.1). Meanwhile Adolphson [1] showed that even when $\mathbb{C}[\mathbb{N}A]$ is not Cohen-Macaulay, the holonomic rank equals $\mathrm{vol}(A)$, as long as $\beta$ is generic in a certain sense. After Strumfels and Takayama showed that the holonomic rank can actually be greater than $\mathrm{vol}(A)$ for non-generic parametes $\beta$, Cattani, D'Andrea and Dickenstein showed that if the covex hull of $A$ is a segment, then there exists a rank-jumping parameter whenever $\mathbb{C}[\mathbb{N}A]$ is not a Cohen-Macaulay ring. Saito, who generalized this result by using different methods, showed that there exist rank-jumping parameters for any non-Cohen-Macaulay simplicial semigroup ring $\mathbb{C}[\mathbb{N}A]$. Matusevich [6] showed that, if the toric ideal defined by $A$ is generic in a certain sense and non-Cohen-Macaulay, then there exists a rank-jumping parameter. However, when we fix a parameter $\beta$, it is not well-known how the holonomic rank is described explicitly except when the covex hull of $A$ is simplicial (see [10], Theorem 6.3). In this paper, using combinatorial notion, we provide a rank formula in the case where the rank of $A$ is three.

1.1. **Definition of $A$-hypergeometric system.** Let $A = (a_1, \ldots, a_n) = (a_{ij})$ be a $d \times n$-matrix of rank $d$ with coefficients in $\mathbb{Z}$. Let $\boldsymbol{k}$ be a field of characteristic zero and $\mathbb{N}$ the set of nonnegative integers. We denote the set $\{a_1, \ldots, a_n\}$ by $A$ as well. Let $\mathcal{F}_A$ denote the face lattice of the cone

$$\mathbb{Q}_{\geq 0}A := \left\{ \sum_{j=1}^{n} c_j a_j | \; c_j \in \mathbb{Q}_{\geq 0} \right\}.$$

Let $\mathbb{N}A$ denote the semigroup generated by $A$ and by $\boldsymbol{k}[\mathbb{N}A]$ its semigroup ring contained in the Laurant polynomial ring $\boldsymbol{k}[t_1^{\pm}, \ldots, t_d^{\pm}]$. For a face $\sigma$ in $\mathcal{F}_A$, we denote by $\mathbb{N}(A \cap \sigma)$ the semigroup generated by $A \cap \sigma$, and by $\mathbb{Z}(A \cap \sigma)$ the group generated by

[1] The detailed version of this paper has been submitted for publication elsewhere.

$A \cap \sigma$. When $A \cap \sigma = \emptyset$, we agree that $\mathbb{N}(A \cap \sigma) = \mathbb{Z}(A \cap \sigma) = 0$. We consider the $\boldsymbol{k}$-algebra homomorphism $\phi_A : \boldsymbol{k}[\partial_1, \ldots, \partial_n] \to \boldsymbol{k}[\mathbb{N}A]$ defined by

$$\phi_A\left(\sum_{u \in \mathbb{N}^n} c_u \partial^u\right) := \sum_{u \in \mathbb{N}^n} c_u t^{Au},$$

where $c_u \in \boldsymbol{k}$, $\partial^u := \partial_1^{u_1} \cdots \partial_n^{u_n}$, and $t^{Au} := t_1^{(Au)_1} \cdots t_d^{(Au)_d}$. We denote by $I_A(\partial)$ the kernel of $\phi_A$ and call it the toric ideal of $A$. Since $\phi_A$ is an epimorphism, we have

$$\boldsymbol{k}[\partial]/I_A(\partial) \overset{\bar{\phi_A}}{\simeq} \boldsymbol{k}[\mathbb{N}A].$$

Given a column parameter vector $\beta = {}^t(\beta_1, \ldots, \beta_d) \in \boldsymbol{k}^d$, let $H_A(\beta)$ denote the left ideal of the $n$-th Weyl algebra

$$D = \boldsymbol{k}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

generated by $I_A(\partial)$ and $\sum_{j=1}^n a_{ij}\theta_j - \beta_i$ $(i = 1, \ldots, d)$, where $\theta_j := x_j \partial_j$. We call the quotient $D$-module $M_A(\beta) := D/H_A(\beta)$ the $A$-hypergeometric system with parameter $\beta$. This system was introduced in the late eighties by Gel'fand, Graev, and Zelevinski (see [4]); its systematic study was started by Gel'fand, Zelevinski, and Kapranov (see, e.g. [5]).

1.2. **Known results on the holonomic rank of $M_A(\beta)$.** In this note, we define the holonomik rank of the $A$-hypergeometric system $\mathrm{rank}(M_A(\beta))$ as follows:

$$\mathrm{rank}(M_A(\beta)) := \dim_{\boldsymbol{k}(x)}(\boldsymbol{k}(x) \otimes_{\boldsymbol{k}[x]} M_A(\beta)).$$

Here $\boldsymbol{k}(x) = \boldsymbol{k}(x_1, \ldots, x_n)$ is the field of rational functions. One of the results shown in [5] about the holonomic rank of $A$-hypergeometric system is that $\mathrm{rank}(M_A(\beta)) = \mathrm{vol}(A)$ for any $\beta \in \boldsymbol{k}^d$ when the semigroup ring $\boldsymbol{k}[\mathbb{N}A]$ is a *Cohen-Macaulay* ring. Here $\mathrm{vol}(A)$ means the normalized volume of the convex hull in $\mathbb{Q}^d$ of $A$ and the origin. This equality can fail if $\boldsymbol{k}[\mathbb{N}A]$ is not a Cohen-Macaulay ring. However, even if we drop the assumption that $\boldsymbol{k}[\mathbb{N}A]$ is a Cohen-Macaulay ring, we have

$$\mathrm{rank}(M_A(\beta)) \geq \mathrm{vol}(A)$$

for any $\beta \in \boldsymbol{k}^d$, and the equality holds for *generic* $\beta$. So we write $j_A(\beta)$ for the gap between the holonomic rank and the volume in this talk.

Moreover, in fact, Matusevich, Miller and Walther [7] completely showed that the rank of $M_A(\beta)$ is indepent of $\beta$, that is, $j_A(\beta) = 0$ for any $\beta$ if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay. However, given a parameter $\beta$, we do not know the formula of the rank of $M_A(\beta)$ except when the convex hull of $A$ is *simplicial*.

## 2. Main Result

2.1. **Combinatorial term $F_A(\beta)$.** As in the previous section, in order to compute the gap $j_A(\beta)$, we introduce a combinatorial term as follows. First, for $\lambda \in \mathbb{Z}A$ and $\beta$, we define the subset $\mathcal{J}(\lambda; \beta)$ of $\mathcal{F}_A$ by

$$\mathcal{J}(\lambda; \beta) := \{\sigma \in \mathcal{F}_A |\ \lambda \notin \mathbb{N}A + \mathbb{Z}(A \cap \sigma), \beta - \lambda \in \boldsymbol{k}(A \cap \sigma)\}.$$

Second, we define a preorder on $\mathbb{Z}A$ as follows:

$$\lambda < \mu \underset{\text{def}}{\iff} \text{for any } \sigma \in \mathcal{J}(\lambda; \beta), \lambda + \mathbb{Z}(A \cap \sigma) = \mu + \mathbb{Z}(A \cap \sigma).$$

Then we have the following proposition on this set.

**Proposition 2.1.** (1) *The set $\mathcal{J}(\lambda;\beta)$ does not contain $\mathbb{Q}_{\geq 0}A$.*
(2) *If $\lambda \in \mathbb{N}A$, then we have $\mathcal{J}(\lambda;\beta) = \emptyset$.*
(3) *If $\lambda < \mu$, then we have $\mathcal{J}(\lambda;\beta) \subset \mathcal{J}(\mu;\beta)$.*

Now, we consider the subset of $\mathbb{Z}A \setminus \mathbb{N}A$:

$$E_A(\beta) = \{\lambda \in \mathbb{Z}A \setminus \mathbb{N}A |\ \mathcal{J}(\lambda;\beta) \neq \varnothing\}.$$

We denote by $F_A(\beta)$ the inductive limit of the set $(E_A(\beta), <)$ which we regard as an inductive system. In other words, $F_A(\beta)$ coincides with the set of maximal elements in $((\mathbb{Z}A \setminus \mathbb{N}A)/\sim, <)$, where $\sim$ means the equivalence relation defined by

$$\lambda \sim \mu \underset{\text{def}}{\iff} \lambda < \mu \text{ and } \lambda > \mu.$$

Since $\lambda \in \beta + \bigcup_{\tau \in \mathcal{F}_A} \boldsymbol{k}(A \cap \tau)$ for any $\lambda \in F_A(\beta)$ and $[\mathbb{Z}A \cap \mathbb{Q}\tau : \mathbb{Z}(A \cap \tau)] < \infty$ for any face $\tau$, we see that $F_A(\beta)$ is a finite set.

2.2. **Main result.** Let $d = 3$ to the end of this note. First assume that the cardinality of $F_A(\beta)$ is one. Let $F_A(\beta) = \{\lambda\}$ and $\widetilde{\mathcal{J}}(\lambda;\beta)$ denote the set of maximal elements in $\mathcal{J}(\lambda;\beta)$. Then the sets $\widetilde{\mathcal{J}}(\lambda;\beta)$ can be classified into four cases:

**(1):** $\widetilde{\mathcal{J}}(\lambda;\beta) = \emptyset$,
**(2):** $\widetilde{\mathcal{J}}(\lambda;\beta)$ consists of one proper face $\sigma$,
**(3):** $\widetilde{\mathcal{J}}(\lambda;\beta)$ consists of all facets,
**(4):** none of the above.

For each case, we have the following theorem:

**Theorem 2.2.** *Let $d = 3$. Assume that the cardinality of $F_A(\beta)$ is one. Then we have the following.*
(1) *$\widetilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (1) $\Rightarrow j_A(\beta) = 0$,*
(2) *$\widetilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (2) $\Rightarrow$*

$$j_A(\beta) = \begin{cases} 0 & \text{if } \sigma \text{ is a facet,} \\ \mathrm{vol}(A \cap \sigma) & \text{if } \sigma \text{ is an edge,} \\ 2 & \text{if } \sigma = \{0\}, \end{cases}$$

(3) *$\widetilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (3) $\Rightarrow j_A(\beta) = 0$,*
(4) *$\widetilde{\mathcal{J}}(\lambda;\beta)$ satisfies the case (4) $\Rightarrow j_A(\beta) = \sum_{\sigma \in \widetilde{\mathcal{J}}(\lambda;\beta):edges} (\mathrm{vol}(A \cap \sigma)) + m - 1$. Here $m$ means the number of connected components of the finite graph $G_\lambda = \{\sigma \in \mathcal{F}_A |\ \{0\} \neq \sigma \subset \tau$ for some $\tau \in \widetilde{\mathcal{J}}(\lambda;\beta)\}$ with respect to the inclusion relation.*

Second not assume that the cardinality of $F_A(\beta)$ is one. In this case, it suffices that for each $\lambda \in F_A(\beta)$ we compute the number determind by the previous theorem, that is, we compute the right hand side of the equality in the theorem, regarding $F_A(\beta)$ as the singleton set $\{\lambda\}$. For each $\lambda \in F_A(\beta)$, let $l_\lambda$ denote the number in this meaning. Then we have the rank formula as desired:

**Theorem 2.3.** *Let $d = 3$. Then we have $j_A(\beta) = \sum_{\lambda \in F_A(\beta)} l_\lambda$.*

## 3. Examples

**Example 1.** Let $A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Then we have $\mathrm{vol}(A_1) = 7$.

First we consider the case where $\beta = {}^t(1,2,0)$. Then we have $F_{A_1}(\beta) = \{\beta\}$ and $\widetilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geqq 0}a_1, \mathbb{Q}_{\geqq 0}a_4\}$. Hence we have $j_{A_1}(\beta) = 1$.

Second we consider the case where $\beta = {}^t(2/5,1,0)$. Then we have $F_{A_1}(\beta) = \{{}^t(1,1,0), {}^t(1,4,0)\}$ and $\widetilde{\mathcal{J}}({}^t(1,1,0);\beta) = \{\mathbb{Q}_{\geqq 0}a_1\}$ and $\widetilde{\mathcal{J}}({}^t(1,4,0);\beta) = \{\mathbb{Q}_{\geqq 0}a_4\}$. Hence $F_{A_1}(\beta)$ is semisimple. Since $\mathbb{Q}_{\geqq 0}a_1$ and $\mathbb{Q}_{\geqq 0}a_4$ are both edges, we have $j_{A_1}(\beta) = 1 + 1 = 2$.

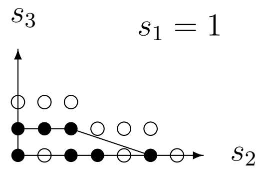


Figure 1. The set $A_1$

**Example 2.** Let $A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}$. Then we have $\mathrm{vol}(A_2) = 6$.

Let $\beta = {}^t(1,1,1)$. Then we have $F_{A_2}(\beta) = \{\beta\}$ and $\widetilde{\mathcal{J}}(\beta;\beta) = \{\{0\}\}$. Hence we have $j_{A_2}(\beta) = 2$.

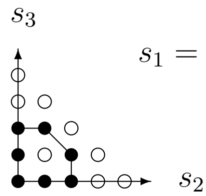


Figure 2. The set $A_2$

**Example 3.** Let $A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$. Then we have $\mathrm{vol}(A_3) = 6$.

Let $\beta = {}^t(0,1,0)$. Then we have $F_{A_3}(\beta) = \{\beta\}$ and $\widetilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geqq 0}a_1 + \mathbb{Q}_{\geqq 0}a_4, \mathbb{Q}_{\geqq 0}a_3 + \mathbb{Q}_{\geqq 0}a_6\}$. Hence we have $j_{A_3}(\beta) = 1$.

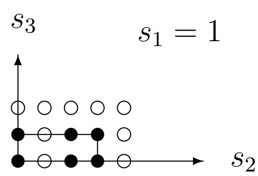


Figure 3. The set $A_3$

**Example 4.** Let $A_4 = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$; not homogeneous. Then we have $\mathrm{vol}(A_4) = 12$.

First we consider the case where $\beta = {}^t(1,1,0)$. Then we have $F_{A_4}(\beta) = \{\beta\}$ and $\widetilde{\mathcal{J}}(\beta;\beta) = \{\mathbb{Q}_{\geqq 0}a_1\}$. Hence we have $j_{A_4}(\beta) = \mathrm{vol}(A_4 \cap \mathbb{Q}_{\geqq 0}a_1) = 3$.

Second we consider the case where $\beta = {}^t(2,2,0)$. Then $F_{A_4}(\beta) = \{\beta\}$ and $\widetilde{\mathcal{J}}(\beta;\beta) = \{\{0\}\}$. Hence we have $j_{A_4}(\beta) = 2$.

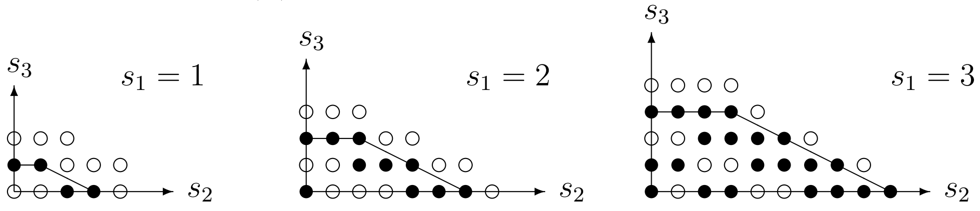


Figure 4. The set $A_4$

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Department of Mathematics
Hokkaido University
Sapporo, 060-0810 JAPAN
*E-mail address*: go_zoku@lapis.plala.or.jp