THE HOLONOMIC RANK FORMULA FOR \(A\)-HYPERGEOMETRIC SYSTEM

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1. Introduction

Given a finite set \(A\) of \(d\)-dimensional integral vectors which belong to one hyperplane off the origin in \(\mathbb{Q}A\) and a parameter vector \(\beta \in \mathbb{C}^d\), Gel’fand, Kapranov and Zelevinskii [5] defined a system of differential equations, called an \(A\)-hypergeometric system \(M_A(\beta)\). They proved that the holonomic rank of \(M_A(\beta)\) equals the normalized volume of the convex hull of \(A\) and the origin (denote by \(\text{vol}(A)\)) for any \(\beta\) when the semigroup ring \(\mathbb{C}[NA]\) determined by \(A\) is Cohen-Macaulay. In general, the rank is not less than the volume (see [1], [13], Theorem 3.5.1). Meanwhile Adolphson [1] showed that even when \(\mathbb{C}[NA]\) is not Cohen-Macaulay, the holonomic rank equals \(\text{vol}(A)\), as long as \(\beta\) is generic in a certain sense. After Strumfels and Takayama showed that the holonomic rank can actually be greater than \(\text{vol}(A)\) for non-generic parameters \(\beta\), Cattani, D’Andrea and Dickenstein showed that if the convex hull of \(A\) is a segment, then there exists a rank-jumping parameter whenever \(\mathbb{C}[NA]\) is not a Cohen-Macaulay ring. Saito, who generalized this result by using different methods, showed that there exist rank-jumping parameters for any non-Cohen-Macaulay simplicial semigroup ring \(\mathbb{C}[NA]\). Matusevich [6] showed that, if the toric ideal defined by \(A\) is generic in a certain sense and non-Cohen-Macaulay, then there exists a rank-jumping parameter. However, when we fix a parameter \(\beta\), it is not well-known how the holonomic rank is described explicitly except when the convex hull of \(A\) is simplicial (see [10], Theorem 6.3). In this paper, using combinatorial notion, we provide a rank formula in the case where the rank of \(A\) is three.

1.1. Definition of \(A\)-hypergeometric system. Let \(A = (a_1, \ldots, a_n) = (a_{ij})\) be a \(d \times n\)-matrix of rank \(d\) with coefficients in \(\mathbb{Z}\). Let \(k\) be a field of characteristic zero and \(N\) the set of nonnegative integers. We denote the set \(\{a_1, \ldots, a_n\}\) by \(A\) as well. Let \(\mathcal{F}_A\) denote the face lattice of the cone

\[
\mathbb{Q}_{\geq 0}A := \left\{ \sum_{j=1}^{n} c_j a_j \mid c_j \in \mathbb{Q}_{\geq 0} \right\}.
\]

Let \(NA\) denote the semigroup generated by \(A\) and by \(k[NA]\) its semigroup ring contained in the Laurent polynomial ring \(k[t_1^\pm, \ldots, t_d^\pm]\). For a face \(\sigma\) in \(\mathcal{F}_A\), we denote by \(N(A \cap \sigma)\) the semigroup generated by \(A \cap \sigma\), and by \(\mathbb{Z}(A \cap \sigma)\) the group generated by

\[\text{Vol}(A)\]
A \cap \sigma$. When $A \cap \sigma = \emptyset$, we agree that $\mathbb{N}(A \cap \sigma) = \mathbb{Z}(A \cap \sigma) = 0$. We consider the $k$-algebra homomorphism $\phi_A : k[\partial, \ldots, \partial_n] \rightarrow k[\mathbb{N}A]$ defined by

$$\phi_A \left( \sum_{n \in \mathbb{N}^n} c_n \partial^n \right) := \sum_{n \in \mathbb{N}^n} c_n t^{A_n},$$

where $c_n \in k$, $\partial^n := \partial_1^{n_1} \cdots \partial_n^{n_n}$, and $t^{A_n} := t_1^{(A_n)} \cdots t_d^{(A_n)}$. We denote by $I_A(\partial)$ the kernel of $\phi_A$ and call it the toric ideal of $A$. Since $\phi_A$ is an epimorphism, we have

$$k[\partial]/I_A(\partial) \cong k[\mathbb{N}A].$$

Given a column parameter vector $\beta = (\beta_1, \ldots, \beta_d) \in k^d$, let $H_A(\beta)$ denote the left ideal of the $n$-th Weyl algebra

$$D = k(x_1, \ldots, x_n, \partial_1, \ldots, \partial_n)$$

generated by $I_A(\partial)$ and $\sum_{j=1}^n a_j \theta_j - \beta_i$ $(i = 1, \ldots, d)$, where $\theta_j := x_j \partial_j$. We call the quotient $D$-module $M_A(\beta) := D/H_A(\beta)$ the $A$-hypergeometric system with parameter $\beta$. This system was introduced in the late eighties by Gel’fand, Graev, and Zelevinski (see [4]); its systematic study was started by Gel’fand, Zelevinski, and Kapranov (see, e.g. [5]).

1.2. Known results on the holonomic rank of $M_A(\beta)$. In this note, we define the holonomik rank of the $A$-hypergeometric system $\text{rank}(M_A(\beta))$ as follows:

$$\text{rank}(M_A(\beta)) := \dim_{k(x)}(k(x) \otimes_{k[x]} M_A(\beta)).$$

Here $k(x) = k(x_1, \ldots, x_n)$ is the field of rational functions. One of the results shown in [5] about the holonomic rank of $A$-hypergeometric system is that $\text{rank}(M_A(\beta)) = \text{vol}(A)$ for any $\beta \in k^d$ when the semigroup ring $k[\mathbb{N}A]$ is a Cohen-Macaulay ring. Here $\text{vol}(A)$ means the normalized volume of the convex hull in $\mathbb{Q}^d$ of $A$ and the origin. This equality can fail if $k[\mathbb{N}A]$ is not a Cohen-Macaulay ring. However, even if we drop the assumption that $k[\mathbb{N}A]$ is a Cohen-Macaulay ring, we have

$$\text{rank}(M_A(\beta)) \geq \text{vol}(A)$$

for any $\beta \in k^d$, and the equality holds for generic $\beta$. So we write $j_A(\beta)$ for the gap between the holonomic rank and the volume in this talk.

Moreover, in fact, Matuschevich, Miller and Walther [7] completely showed that the rank of $M_A(\beta)$ is indepent of $\beta$, that is, $j_A(\beta) = 0$ for any $\beta$ if and only if $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay. However, given a parameter $\beta$, we do not know the formula of the rank of $M_A(\beta)$ except when the convex hull of $A$ is simplicial.

2. Main Result

2.1. Combinatorial term $F_A(\beta)$. As in the previous section, in order to compute the gap $j_A(\beta)$, we introduce a combinatorial term as follows. First, for $\lambda \in \mathbb{Z}A$ and $\beta$, we define the subset $\mathcal{J}(\lambda; \beta)$ of $\mathcal{F}_A$ by

$$\mathcal{J}(\lambda; \beta) := \{ \sigma \in \mathcal{F}_A \mid \lambda \notin \mathbb{N}A + \mathbb{Z}(A \cap \sigma), \beta - \lambda \in k(A \cap \sigma) \}. $$

Second, we define a preorder on $\mathbb{Z}A$ as follows:

$$\lambda < \mu \overset{\text{def}}{\iff} \text{for any } \sigma \in \mathcal{J}(\lambda; \beta), \lambda + \mathbb{Z}(A \cap \sigma) = \mu + \mathbb{Z}(A \cap \sigma).$$

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Then we have the following proposition on this set.

**Proposition 2.1.** (1) The set $J(\lambda; \beta)$ does not contain $Q_{\geq 0} A$.

(2) If $\lambda \in NA$, then we have $J(\lambda; \beta) = \emptyset$.

(3) If $\lambda < \mu$, then we have $J(\lambda; \beta) \subset J(\mu; \beta)$.

Now, we consider the subset of $ZA \setminus NA$:

$$E_A(\beta) = \{\lambda \in ZA \setminus NA | J(\lambda; \beta) \neq \emptyset\}.$$  

We denote by $F_A(\beta)$ the inductive limit of the set $(E_A(\beta), <)$ which we regard as an inductive system. In other words, $F_A(\beta)$ coincides with the set of maximal elements in $((ZA \setminus NA)/ \sim, <)$, where $\sim$ means the equivalence relation defined by

$$\lambda \sim \mu \iff \lambda < \mu \text{ and } \lambda > \mu.$$  

Since $\lambda \in \beta + \bigcup_{\tau \in F_A} k(A \cap \tau)$ for any $\lambda \in F_A(\beta)$ and $|ZA \cap Q\tau : Z(A \cap \tau)| < \infty$ for any face $\tau$, we see that $F_A(\beta)$ is a finite set.

2.2. **Main result.** Let $d = 3$ to the end of this note. First assume that the cardinality of $F_A(\beta)$ is one. Let $F_A(\beta) = \{\lambda\}$ and $\tilde{J}(\lambda; \beta)$ denote the set of maximal elements in $J(\lambda; \beta)$. Then the sets $J(\lambda; \beta)$ can be classified into four cases:

(1): $\tilde{J}(\lambda; \beta) = \emptyset$,

(2): $\tilde{J}(\lambda; \beta)$ consists of one proper face $\sigma$,

(3): $\tilde{J}(\lambda; \beta)$ consists of all facets,

(4): none of the above.

For each case, we have the following theorem:

**Theorem 2.2.** Let $d = 3$. Assume that the cardinality of $F_A(\beta)$ is one. Then we have the following:

(1) $\tilde{J}(\lambda; \beta)$ satisfies the case (1) $\Rightarrow j_A(\beta) = 0$,

(2) $\tilde{J}(\lambda; \beta)$ satisfies the case (2) $\Rightarrow j_A(\beta) = 0$,

$$j_A(\beta) = \begin{cases} 
0 & \text{if } \sigma \text{ is a facet}, \\
\text{vol}(A \cap \sigma) & \text{if } \sigma \text{ is an edge}, \\
2 & \text{if } \sigma = \{0\},
\end{cases}$$

(3) $\tilde{J}(\lambda; \beta)$ satisfies the case (3) $\Rightarrow j_A(\beta) = 0$,

(4) $\tilde{J}(\lambda; \beta)$ satisfies the case (4) $\Rightarrow j_A(\beta) = \sum_{\sigma \in \tilde{J}(\lambda; \beta) \setminus \text{edges}} (\text{vol}(A \cap \sigma)) + m - 1$.

Here $m$ means the number of connected components of the finite graph $G_{\lambda} = \{\sigma \in F_A | \{0\} \neq \sigma \subset \tau \text{ for some } \tau \in \tilde{J}(\lambda; \beta)\}$ with respect to the inclusion relation.

Second not assume that the cardinality of $F_A(\beta)$ is one. In this case, it suffices that for each $\lambda \in F_A(\beta)$ we compute the number determined by the previous theorem, that is, we compute the right hand side of the equality in the theorem, regarding $F_A(\beta)$ as the singleton set $\{\lambda\}$. For each $\lambda \in F_A(\beta)$, let $l_\lambda$ denote the number in this meaning. Then we have the rank formula as desired:

**Theorem 2.3.** Let $d = 3$. Then we have $j_A(\beta) = \sum_{\lambda \in F_A(\beta)} l_\lambda$. 

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3. Examples

Example 1. Let $A_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$. Then we have $\text{vol}(A_1) = 7$.

First we consider the case where $\beta = \tau(1, 2, 0)$. Then we have $F_{A_1}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta; \beta) = \{Q_{\geq 0} a_1, Q_{\geq 0} a_4\}$. Hence we have $j_{A_1}(\beta) = 1$.

Second we consider the case where $\beta = \tau(2/5, 1, 0)$. Then we have $F_{A_1}(\beta) = \{\tau(1, 1, 0), \tau(1, 4, 0)\}$ and $\tilde{\mathcal{J}}(\tau(1, 1, 0); \beta) = \{Q_{\geq 0} a_1\}$ and $\tilde{\mathcal{J}}(\tau(1, 4, 0); \beta) = \{Q_{\geq 0} a_4\}$. Hence $F_{A_1}(\beta)$ is semisimple. Since $Q_{\geq 0} a_1$ and $Q_{\geq 0} a_4$ are both edges, we have $j_{A_1}(\beta) = 1 + 1 = 2$.

![Figure 1. The set $A_1$](image)

Example 2. Let $A_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & 2 & 2 \end{pmatrix}$. Then we have $\text{vol}(A_2) = 6$.

Let $\beta = \tau(1, 1, 1)$. Then we have $F_{A_2}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta; \beta) = \{\{0\}\}$. Hence we have $j_{A_2}(\beta) = 2$.

![Figure 2. The set $A_2$](image)

Example 3. Let $A_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$. Then we have $\text{vol}(A_3) = 6$.

Let $\beta = \tau(0, 1, 0)$. Then we have $F_{A_3}(\beta) = \{\beta\}$ and $\tilde{\mathcal{J}}(\beta; \beta) = \{Q_{\geq 0} a_1 + Q_{\geq 0} a_4, Q_{\geq 0} a_3 + Q_{\geq 0} a_6\}$. Hence we have $j_{A_3}(\beta) = 1$.

![Figure 3. The set $A_3$](image)
Example 4. Let $A_4 = \begin{pmatrix} 3 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$; not homogeneous. Then we have $\text{vol}(A_4) = 12$.

First we consider the case where $\beta = t(1, 1, 0)$. Then we have $F_{A_4}(\beta) = \{\beta\}$ and $\mathcal{F}(\beta; \beta) = \{Q_{\geq 0} a_1\}$. Hence we have $j_{A_4}(\beta) = \text{vol}(A_4 \cap Q_{\geq 0} a_1) = 3$.

Second we consider the case where $\beta = t(2, 2, 0)$. Then $F_{A_4}(\beta) = \{\beta\}$ and $\mathcal{F}(\beta; \beta) = \{\{0\}\}$. Hence we have $j_{A_4}(\beta) = 2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The set $A_4$}
\end{figure}

References