# STABLE EQUIVALENCES INDUCED FROM GENERALIZED TILTING MODULES II [1]

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## 1. Introduction

Let $A$ and $B$ be finite dimensional algebras over a field $K$. We suppose that ${}_BT_A$ is a generalized tilting module and $(\varphi, \psi)$ an admissible system for a symmetric algebra, where ${}_AM \otimes_A M_A \stackrel{\varphi}{\to} {}_AM_A$ and ${}_AM \otimes_A M_A \stackrel{\psi}{\to} {}_A\mathrm{D}A_A$. Then the transformed system $(\varphi^T, \psi^T)$ is defined over the bimodule ${}_BM_B^T = {}_BT \otimes_A \mathrm{Hom}_A(T, M)_B$ and we have two symmetric algebras $\Lambda(\varphi, \psi) = A \oplus M \oplus \mathrm{D}A$ and $\Lambda(\varphi^T, \psi^T) = B \oplus M^T \oplus \mathrm{D}B$ under the assumption (1) the canonical map ${}_BT \otimes_A \mathrm{Hom}_A(T, M)_B \stackrel{\theta}{\to} {}_B\mathrm{Hom}_A(T, T \otimes_A M)_B$ defined by $\theta(t \otimes f)(t') = t \otimes f(t')$ is bijective. In the previous note[2], we have shown the existence of a stable equivalence

$$\mathcal{S} : \underline{\mathrm{mod}} - \Lambda(\varphi, \psi) \;\approx\; \underline{\mathrm{mod}} - \Lambda(\varphi^T, \psi^T)$$

by using the assumptions (2) the class $\mathcal{C}(T_A) = \mathrm{gen}^*(T_A) \cap \bigcap_{n>0} \mathrm{KerExt}_A^n(T, ?)$ is contravariantly finite in mod$-A$ and, dually, $\mathcal{D}(\mathrm{D}T_B) = \mathrm{cog}^*(\mathrm{D}T_B) \cap \bigcap_{n>0} \mathrm{KerExt}_B^n(?, \mathrm{D}T)$ covariantly finite in mod$-B$, and (3) the modules $M_A$ and $T \otimes_A M_A$ are in the class $\mathcal{C}(T_A)$. Those assumptions (1) to (3) are satisfied if we suppose

(a) the module ${}_AM_A$ is of the form $\bigoplus_{(X,Y)} {}_AX \otimes_K Y_A$ with all $Y_A$'s are in the class $\mathcal{C}(T_A)$, and

(b) one of the algebras $A$ and $B$ is representation-finite.

The purpose of the present note is to give an example of a couple of an admissible system $(\varphi, \psi)$ and a generalized tilting module ${}_BT_A$ for which the symmetric algebras $\Lambda(\varphi, \psi)$ and $\Lambda(\varphi^T, \psi^T)$ are stably equivalent but not derived equivalent. Such an example means that our stable equivalence $\mathcal{S}$ is not induced from Morita theory of derived categories.

## 2. An Example

Define an algebra $A$ by the quiver

$$\mathrm{Q}(A) : \quad \begin{array}{ccc} & \beta & \\ 1 & \to & 2 \\ \circlearrowleft & & \circlearrowright \\ \alpha & & \gamma \end{array}$$

with the relations $\alpha^2 = 0$, $\gamma^2 = 0$, $\beta \cdot \alpha = 0$ and $\gamma \cdot \beta = 0$. It is checked that the algebra $A$ is representation-finite with only eight non-isomorphic indecomposable modules. We also

[1] The detailed version of this paper will be submitted for publication elsewhere.

have

$$A_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} & 2 & \\ 1 & & 2 \end{pmatrix} \quad \text{and} \quad \mathrm{D}A_A = \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Choose a generalized tilting module as $T_A = \begin{pmatrix} 1 & & 2 & \\ & 1 & & 2 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}$. Then, the quiver $\mathrm{Q}(B)$ of $B = \mathrm{End}(T_A)$ is given by $\underset{\circlearrowleft}{1} \leftrightarrows 2$ and we have

$$B_B = \begin{pmatrix} & 1 & & \\ 1 & & 2 & \\ & & & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathrm{D}B_B = \begin{pmatrix} 2 & & & & \\ & 1 & & & \\ & & 2 & & 1 \\ & & & 1 & \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Now, we set ${}_AM_A = {}_AAe_1 \otimes_K e_1\mathrm{D}A_A = \begin{pmatrix} & 1 & \\ 1 & & 2 \end{pmatrix} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix}$ and $\varphi = 0$ the zero map from ${}_AM \otimes_A M_A$ to ${}_AM_A$. Since our module ${}_AM_A$ is canonically isomorphic to its dual ${}_A\mathrm{D}M_A$, we have a map $\psi : {}_AM \otimes_A M_A \to {}_A\mathrm{D}A_A$ and $(\varphi, \psi)$ becomes an admissible system for a symmetric algebra. Then, the assumptions (a) and (b) are satisfied and, therefore, the symmetric algebras $\Lambda = \Lambda(\varphi, \psi)$ and $\Gamma = \Lambda(\varphi^T, \psi^T)$ are stably equivalent.

In order to prove that the algebras $\Lambda$ and $\Gamma$ are not derived equivalent, we use the following well-known result. The proof can be seen in the paper [1] by Usami.

**Lemma 1.** *If the algebras* $\Lambda$ *and* $\Gamma$ *are derived equivalent, there exists a regular matrix* $P \in \mathrm{Mat}_n(\mathbb{Z})$ *and their Cartan matrices satisfy the equation* ${}^tP \cdot \mathrm{C}_\Lambda \cdot P = \mathrm{C}_\Gamma$.

We have $\mathrm{C}_\Lambda = \begin{pmatrix} 8 & 3 \\ 3 & 5 \end{pmatrix}$ since

$$e_1\Lambda_A = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \right\} \oplus \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix}$$

and

$$e_2\Lambda_A = \begin{pmatrix} & 2 & \\ 1 & & 2 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 & & 2 \\ & 1 & \end{pmatrix} \right\} \oplus \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Similarly, we have $\mathrm{C}_\Gamma = \begin{pmatrix} 10 & 3 \\ 3 & 4 \end{pmatrix}$ from

$$f_1\Gamma_B = \begin{pmatrix} & 1 & & \\ 1 & & 2 & \\ & & & 1 \end{pmatrix} \oplus \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \oplus \begin{pmatrix} & & & & 2 \\ & & & 1 & \\ 1 & & 2 & & \\ & 1 & & & \end{pmatrix}$$

and

$$f_2\Gamma_B = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} \oplus \{ 0 \} \oplus \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},$$

where $e_i$ (resp. $f_i$) stands for the primitive idempotent element in the algebra $A$ (resp. $B$) corresponding to the vertex in $\mathrm{Q}(A)$ (resp. $\mathrm{Q}(B)$) indexed by $i$ and $n$ is the common number of non-isomorphic simple $A$- or $B$-modules.

Put $P = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z})$, then we have

$$ {}^tP \cdot \mathrm{C}_\Lambda \cdot P = \begin{pmatrix} 8a^2 + 6ab + 5b^2 & 8ac + 3(ad + bc) + 5bd \\ 8ac + 3(ad + bc) + 5bd & 8c^2 + 6cd + 5d^2 \end{pmatrix}. $$

Hence, ${}^tP \cdot \mathrm{C}_\Lambda \cdot P = \mathrm{C}_\Gamma$ implies that

$$ 5c^2 + 3(c + d)^2 + 2d^2 = 8c^2 + 6cd + 5d^2 = 4, $$

and this is impossible for integers $c, d \in \mathbb{Z}$. Therefore, the algebras $\Lambda$ and $\Gamma$ are not derived equivalent by the previous lemma.

## References


[1] Y. Usami, *Derived equivalence and perfect isometry II*, Proceedings of the 4th Symposium on Representation Theory of Algebras, 165-189, Saitama, 1994. (in Japanese)

[2] T. Wakamatsu, *Stable equivalences induced from generalized tilting modules*, Proceedinds of the 36th Symposium on Ring Theory and Representation Theory, 147-156, Yamanashi, 2004.



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