STABILITY OF FROBENIUS ALGEBRAS WITH POSITIVE GALOIS COVERINGS [1]

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ABSTRACT. A finite dimensional self-injective algebra will be determined when it is stably equivalent to a positive self-injective algebra of Dynkin, tilted or quasi-tilted type.

1. INTRODUCTION

In a series of joint work with A. Skowroński, we have been studying self-injective algebras with Galois coverings by repetitive algebras [10] [11] [12] [13], and [4]. One of the aims of the work is to characterize self-injective algebras ring theoretically or module categorically which have Galois coverings by the repetitive algebras. A ring theoretical criterion theorem was found in [11] for a self-injective algebra to have a Galois covering by a repetitive algebra. On the other hand, study of the module category over a self-injective algebra depends on the type of a repetitive algebra which defines a Galois covering. In this survey paper, some of main theorems related to module categories in the joint work are arranged into two theorems (Theorems 2, 3). The proofs, however, are not unified (and refer the proofs in the references mentioned above), but a common idea of the proofs is to find an ideal so that a criterion theorem is applicable to conclude the existence of a Galois covering by a repetitive algebra.

Throughout this paper, K will be a fixed (commutative) field, and by an algebra we mean a basic and associative K-algebra which is not necessarily finite dimensional, but with a complete set of orthogonal primitive idempotents, that is, with a set of orthogonal primitive idempotents, say $\{e_i\}_{i\in I}$ of an algebra Λ, such that $\Lambda = \bigoplus_{i\in I} \Lambda e_i = \bigoplus_{i\in I} e_i\Lambda$. A K-category R is a category whose hom-sets are K-spaces and composition of morphisms are K-bilinear. Then, K-categories (finite K-categories, respectively) are in one-to-one correspondence with algebras having fixed complete set of orthogonal primitive idempotents (finite dimensional algebras with identity, respectively). We freely identify K-categories with K-algebras. A will be a finite dimensional, connected *self-injective* algebra, and B and Λ will be finite dimensional algebras. Two finite dimensional self-injective algebras A_1, A_2 are said to be *socle equivalent* if $A_1/\operatorname{soc} A_1 \cong A_2/\operatorname{soc} A_2$ where soc is the socle of an algebra, and *stably equivalent* if their stable categories $\underline{\mathrm{mod}}A_1$ and $\underline{\mathrm{mod}}A_2$ of the categories of finite dimensional modules are isomorphic.

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2. Repetitive categories

2.1. Let B be a K-category, and $B_n = B$, $DB_n = DB$ the copies of B and DB, respectively. The *repetitive category* $\widehat{B}$ of B is the direct sum of K-spaces $\widehat{B} = \bigoplus_{n\in\mathbb{Z}}(B_n \oplus DB_n)$ with multiplication:

$$\big(\textstyle\sum_i (b_i, f_i)\big) \cdot \big(\textstyle\sum_j (c_j, g_j)\big) = \textstyle\sum_i (b_i c_i, b_i g_i + f_i c_{i+1})$$

for $b_i, c_j \in B_i$ and $f_i, g_j \in DB_i$ for $i \in \mathbb{Z}$. The category may be written as a matrix algebra without identity in the following way:

$$\widehat{B} = \begin{pmatrix} \ddots & \ddots & & & 0 \\ & B_{n-1} & DB_{n-1} & & \\ & & B_n & DB_n & \\ & & & B_{n+1} & \ddots \\ 0 & & & & \ddots \end{pmatrix}$$

where all B_n lie on the diagonal, and $\widehat{B}$ is the set of those matrices that all but finitely many entries are zero. Summation and multiplication are defined as ones of matrices with $DB \otimes DB \to 0$ zero map. $\widehat{B}$ is a K-category whose object set is the disjoint union of copies of $Obj(B)$ in each B_n.

2.2. A group G of automorphisms of a K-category $\widehat{B}$ is said to be *admissible* if any automorphism of G acts freely on $Obj(\widehat{B})$, and the G-orbit sets of $Obj(\widehat{B})$ is finite. The category $\widehat{B}/G$ is then naturally defined by the G-orbits of $\widehat{B}$-objects and $\widehat{B}$-morphisms. See [1] for details. The orbit category $\widehat{B}/G$ is clearly finite dimensional and self-injective.

An automorphism $\nu_{\widehat{B}}$ of $\widehat{B}$ is called the *Nakayama automorphism* of $\widehat{B}$ if the restriction to any $B_n \oplus DB_n$ is identity onto $B_{n+1} \oplus DB_{n+1}$. An automorphism φ of $\widehat{B}$ is said to be *positive* if $\varphi(B_n) \subseteq \sum_{i\geq n}(B_i \oplus DB_i)$ $(n \in \mathbb{Z})$ or, equivalently, $\varphi(Obj(B_n)) \subseteq \bigcup_{i\geq n} Obj(B_i)$ $(n \in \mathbb{Z})$, and *strictly positive* if φ is positive and $\varphi(B_n) \neq B_n$ for all n.

Importance of the orbit categories by repetitive categories may be suggested by the classification theorem of representation-finite self-injective algebras over an algebraically closed field, which was proved by C. Riedtmann [8] [9] (and also by D. Hughes - J. Waschbüsch [3]). Also, D. Happel (1991) showed that the bounded category $D^b(\mathrm{mod}\, B)$ of a finite dimensional algebra B is embedded into the stable category $\underline{\mathrm{mod}}\widehat{B}$ of the category of finite dimensional modules over $\widehat{B}$. Moreover, $D^b(\mathrm{mod}\, B)$ is isomorphic to $\underline{\mathrm{mod}}\widehat{B}$ if and only if $\mathrm{gldim}\, B < \infty$.

2.3. (1) A typical example of the orbit algebras by repetitive algebras is $\widehat{B}/\langle \nu_{\widehat{B}} \rangle$ which is isomorphic to the trivial extension algebra $B \ltimes DB$.

(2) Let A be 4-dimensional, local and self-injective algebras such that

$$A \cong \widehat{B}/\langle \varphi \rangle;\ B = k[x]/(x^2),\ \varphi^2 = \psi \nu_{\widehat{B}}$$

for a positive automorphisms φ, ψ, and they are socle equivalent. Those algebras were first presented by T. Nakayama and C. Nesbitt in 1938 [5]. J. Rickard showed that such algebras are isomorphic if they are stably equivalent (see [15] [6]). Thus it should be noted that a socle equivalence does not imply a stable equivalence, in general.

3. Type of self-injective algebras

3.1. An algebra B is called *quasi-tilted algebra* if $B \cong \mathrm{End}_{\mathcal{H}}(T)$ where T is a tilting object in a hereditary abelian K-category $\mathcal{H}$, that is, $\mathrm{Hom}_{\mathcal{H}}(X,Y)$ and $\mathrm{Ext}^1_{\mathcal{H}}(X,Y)$ are finite dimensional and $\mathrm{Ext}^2_{\mathcal{H}}(X,Y) = 0$ for all objects X, Y of $\mathcal{H}$. This is equivalent to the statement that $\mathrm{gldim}(B) \leq 2$ and every finitely generated indecomposable B-module X has $\mathrm{pd}(X) \leq 1$ or $\mathrm{id}(X) \leq 1$. An algebra is said to be *canonical* [7] if its ordinary quiver has a unique source ω and a unique sink o, and consists of paths $p_1, \ldots, p_n$ $(n \geq 2)$ from ω to o, which meet each other only at ω and o. Moreover, in case $n = 2$ there is no non-trivial relation (i.e., the algebra is hereditary), and in case $n \geq 3$ the length of each path is more than 1 and relations are

$$p_1 + p_2 + p_3 = 0, \;\; p_1 + \lambda_i p_2 + p_i = 0 \;\; (i = 4, \ldots, n)$$

for $\lambda_i \neq 0, 1$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$.

Theorem 1. [2] *An algebra B is quasi-tilted if and only if $B \cong \mathrm{End}_{\Lambda}(T)$ where T is a tilting module over a hereditary algebra or a canonical algebra Λ.*

3.2. Let A be a self-injective algebra. A is said to be *positive* (*strictly positive*, respectively) if $A \cong \widehat{B}/\langle \varphi \nu_{\widehat{B}} \rangle$ for some algebra B and *positive* (*strictly positive*, respectively) automorphism φ of $\widehat{B}$. (See [16].) A positive self-injective algebra A is said to be of

(i) (Δ-)*tilted type* if B is a tilted algebra of type Δ,

(ii) *canonical type* if B is the endomorphism algebra of a tilting module over a *canonical* algebra,

(iii) *quasi-tilted type* if B is a quasi-tilted algebra.

Thus, by Theorem 1, a positive self-injective algebra A is of quasi-tilted type if and only if A is of tilted type or canonical type.

4. Main Theorems

4.1. Some of main results on the module categories in [10] [12] [13] [4] are stated as follows.

Theorem 2. *A self-injective algebra A is stably equivalent to a positive self-injective algebra of Δ-tilted type if and only if A is socle equivalent to a positive self-injective algebra of Δ-tilted type. Moreover, in case K is algebraically closed, those statements are equivalent to the statement that A is isomorphic to a positive self-injective algebra of Δ-tilted type.*

Theorem 3. *The followings are equivalent.*

(1) *A is stably equivalent to a strongly positive self-injective algebra of quasi-tilted type.*

(2) *A is isomorphic to a strongly positive self-injective algebra of quasi-tilted type.*

4.2. Theorem 2 was first proved in [10] for algebras of non-Dynkin type, and later in [13] for algebras of Dynkin type ($\Delta = A_n, B_n, C_n, D_n\,(n \geq 4), E_6, E_7, E_8, F_4, G_2$). On the other hand, Theorem 3 was proved in [12] for non-Dynkin type, and recently in [4] for canonical type where a new characterization of a quasi-tilted algebra is given and a precise observation is required for the form of Auslander-Reiten components.

In any case, however, we have to find an ideal I satisfying the annihilator condition $r_A(I) = eI$ for some $e = e^2$, and then, apply the following criterion theorem for the existence of a positive Galois covering.

Criterion Theorem 4 . [11] *A self-injective algebra Λ is positive if there is an ideal I of Λ such that, for some idempotent e of Λ, the following conditions are satisfied:*
(1) $r_\Lambda(I) = eI$,
(2) *The canonical algebra epimorphism $e\Lambda e \to e\Lambda e/eIe$ splits.*
In this case, Λ is isomorphic to $\widehat{B}/\langle\varphi\nu_{\widehat{B}}\rangle$ for $B = \Lambda/I$ and a positive automorphism φ of $\widehat{B}$.

The idea of the application of the criterion theorem is to construct another self-injective algebra Λ by making use of the annihilator condition (1) (see [11]), so that Λ has both conditions (1) and (2), and A and Λ are socle equivalent. Thus we can know the existence of a positive Galois covering of A *up to socle*. The converse of Theorem 4 is also true [14] and refer to the survey paper [16].

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