HOCHSCHILD COHOMOLOGY OF STRATIFIED ALGEBRAS

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1. Introduction

When studying Hochschild cohomology it is natural to try relating cohomology of an algebra B to that of an 'easier' or 'smaller' algebra A. One such situation is that of B being a one-point extension of A, which has been studied by Happel[5]. More recently, Happel's long exact sequence has been generalized to triangular matrix algebras, for example by Michelena and Platzeck [6], C.Cibils, E.Marcos, M.J.Redondo and A.Solotar [2] and E.L.Green and O.Solberg [4].

We would like to suggest to try further generalizing these results. Natural generalizations of directed or triangular algebras are stratified algebras (when just keeping good homological connections between B and its quotient A).

2. Preliminaries

Let k be a field. Throughout this paper, all algebras are finite dimensional k-algebras and all modules are left modules unless otherwise stated. For any algebra A, we denote by A^e the enveloping algebra $A \otimes_k A^{\mathrm{op}}$. We prepare the following lemma for the next section.

Lemma 1. *Let X be a A-B-bimodule, Y a B-C-bimodule and Z a A-C-bimodule. We have the following isomorphisms:*

(1) *If $\mathrm{Tor}_i^B(X,Y) = 0$ and $\mathrm{Ext}_C^i(Y,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{A-B}(X, \mathrm{Hom}_C(Y,Z)).$$

(2) *If $\mathrm{Tor}_i^B(X,Y) = 0$ and $\mathrm{Ext}_A^i(X,Z) = 0$ for all $i \geq 1$ then, for any $n \geq 0$,*

$$\mathrm{Ext}^n_{A-C}(X \otimes_B Y, Z) \cong \mathrm{Ext}^n_{B-C}(Y, Hom_A(X,Z)).$$

Proof. See Cartan-Eilenberg's book [1]. □

Lemma 2. *Let I be an ideal of an algebra B. If $\mathrm{Ext}^i_{B^e}(I, B/I) = 0$ for all $i \geq 0$, then we have the following two long exact sequences:*

(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B,I) \to \mathrm{Ext}^n_{B^e}(B,B) \to \mathrm{Ext}^n_{B^e}(B/I,B/I) \to \mathrm{Ext}^{n+1}_{B^e}(B,I) \to \cdots$;

(2) $\cdots \to \mathrm{Ext}^n_{B^e}(B/I,I) \to \mathrm{Ext}^n_{B^e}(B,B) \to \mathrm{Ext}^n_{B^e}(B/I,B/I) \oplus \mathrm{Ext}^n_{B^e}(I,I)$
$\to \mathrm{Ext}^{n+1}_{B^e}(B/I,I) \to \cdots$.

The detailed version of this paper will be submitted for publication elsewhere.

Proof. We denote B/I by A, the inclusion $I \to B$ by f and the surjection $B \to A$ by g. We consider the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\mathrm{Ext}^{i-2}_{B^e}(I,A) \arrow[r] \arrow[d] & \mathrm{Ext}^{i-1}_{B^e}(I,I) \arrow[r, "f^I_{i-1}"] \arrow[d, "\delta_I^{i-1}"] & \mathrm{Ext}^{i-1}_{B^e}(I,B) \arrow[r] \arrow[d] & \mathrm{Ext}^{i-1}_{B^e}(I,A) \arrow[d] \\
\mathrm{Ext}^{i-1}_{B^e}(A,A) \arrow[r, "\delta^A_{i-1}"] \arrow[d, "g_A^{i-1}"] & \mathrm{Ext}^{i}_{B^e}(A,I) \arrow[r] \arrow[d, "g_I^i"] & \mathrm{Ext}^{i}_{B^e}(A,B) \arrow[r] \arrow[d] & \mathrm{Ext}^{i}_{B^e}(A,A) \arrow[d, "g_A^i"] \\
\mathrm{Ext}^{i-1}_{B^e}(B,A) \arrow[r, "\delta^B_{i-1}"] \arrow[d] & \mathrm{Ext}^{i}_{B^e}(B,I) \arrow[r, "f_i^B"] \arrow[d] & \mathrm{Ext}^{i}_{B^e}(B,B) \arrow[r, "g_i^B"] \arrow[d, "f_B^i"] & \mathrm{Ext}^{i}_{B^e}(B,A) \arrow[d] \\
\mathrm{Ext}^{i-1}_{B^e}(I,A) \arrow[r] & \mathrm{Ext}^{i}_{B^e}(I,I) \arrow[r, "f_i^I"] & \mathrm{Ext}^{i}_{B^e}(I,B) \arrow[r] & \mathrm{Ext}^{i}_{B^e}(I,A).
\end{tikzcd}$$

Since $\mathrm{Ext}^n(I,A) = 0$ for all $n \geq 0$, we have that f_n^I and g_A^n are isomorphic for all $n \geq 0$. It is not diffecult to show that the followig two sequences are exact:

$$\mathrm{Ext}^{i-1}_{B^e}(A,A) \xrightarrow{\delta^B_{i-1} g_A^{i-1}} \mathrm{Ext}^{i}_{B^e}(B,I) \xrightarrow{f_i^B} \mathrm{Ext}^{i}_{B^e}(B,B) \xrightarrow{(g_A^i)^{-1} g_i^B} \mathrm{Ext}^{i}_{B^e}(A,A)$$

and

$$\mathrm{Ext}^{i-1}_{B^e}(A,A) \oplus \mathrm{Ext}^{i-1}_{B^e}(I,I) \xrightarrow{(-\delta^A_{i-1}, \delta_I^{i-1})} \mathrm{Ext}^{i}_{B^e}(A,I) \xrightarrow{f_i^B g_I^i} \mathrm{Ext}^{i}_{B^e}(B,B)$$

$$\xrightarrow{\begin{pmatrix} (g_A^i)^{-1} g_i^B \\ (f_i^I)^{-1} f_B^i \end{pmatrix}} \mathrm{Ext}^{i}_{B^e}(A,A) \oplus \mathrm{Ext}^{i}_{B^e}(I,I) \xrightarrow{(-\delta_i^A, \delta_I^i)} \mathrm{Ext}^{i+1}_{B^e}(A,I).$$

□

3. Stratifying ideals

Definition 3 (Cline, Parshall and Scott [3], 2.1.1 and 2.1.2)**.** Let B be a finite dimensional algebra and $e = e^2$ an idempotent. The two-sided ideal $J = BeB$ generated by e is called a *stratifying ideal* if the following equivalent conditions (A) and (B) are satisfied:
(A) (a) The multiplication map $Be \otimes_{eBe} eB \to BeB$ is an isomorphism.
(b) For all $n > 0$: $\mathrm{Tor}_n^{eBe}(Be, eB) = 0$.
(B) The epimorphism $B \to A := B/BeB$ induces isomorphisms

$$\mathrm{Ext}^*_A(X,Y) \simeq \mathrm{Ext}^*_B(X,Y)$$

for all A-modules X and Y.

Example 4. (1) Any ideal generated by a central idempotent is a stratifying ideal.
(2) If an algebra A has an idempotent e such that $eA(1-e) = 0$, then AeA and $A(1-e)A$ are both stratifying ideals, namely, triangular matrix algebras have stratifying ideals.
(3) Heredity ideals (used to define quasi-hereditary algebras) are examples of stratifying ideals.

Proposition 5. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. For any $i \geq 0$ and finite dimensional A^e-module M, the induced morphism $g^i_M : \mathrm{Ext}^i_{A^e}(A, M) \to \mathrm{Ext}^i_{B^e}(B, M)$ is isomorphic.*

Proof. It is enough to show that the induced morphisms g^i_M is isomorphic for any simple A^e-module M. For any A-modules X and Y, there exists the following commutative diagram:

$$\begin{array}{ccc}
\mathrm{Ext}^i_{A^e}(A, \mathrm{Hom}_k(X,Y)) & \xrightarrow{\sim} & \mathrm{Ext}^i_A(X,Y) \\
\Big\downarrow{\scriptstyle g^i_{\mathrm{Hom}_k(X,Y)}} & & \Big\downarrow \\
\mathrm{Ext}^i_{B^e}(B, \mathrm{Hom}_k(X,Y)) & \xrightarrow{\sim} & \mathrm{Ext}^i_B(X,Y),
\end{array}$$

where these isomorphisms in rows are induced from Lemma1. Any simple A^e-module has the form of $\mathrm{Hom}_k(X,Y)$ for some simple A-modules X and Y. Hence, by using the condition (B) of stratifying ideals, it is shown that g^i_M is isomorphic for any simple A^e-module M. □

Proposition 6. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. For any $i \geq 0$, the following hold:*

(1) $\mathrm{Ext}^i_{B^e}(BeB, A) = 0$;
(2) $\mathrm{Ext}^i_{B^e}(BeB, BeB) \cong \mathrm{Ext}^i_{eBe^e}(eBe, eBe)$;
(3) $\mathrm{Ext}^i_{B^e}(A, A) \cong \mathrm{Ext}^i_{A^e}(A, A)$.

Proof. By Lemma1 and the condition (A) of stratifying ideals, for any B^e-module X, we have that

$$\begin{aligned}
\mathrm{Ext}^i_{B^e}(BeB, X) &\cong \mathrm{Ext}^i_{B^e}(Be \otimes_{eBe} eB, X) \\
&\cong \mathrm{Ext}^i_{B-eBe}(Be, \mathrm{Hom}_B(eB, X)) \\
&\cong \mathrm{Ext}^i_{B-eBe}(Be \otimes_{eBe} eBe, Xe) \\
&\cong \mathrm{Ext}^i_{eBe^e}(eBe, \mathrm{Hom}_B(Be, Xe)) \\
&\cong \mathrm{Ext}^i_{eBe^e}(eBe, eXe).
\end{aligned}$$

Hence (1) and (2) hold.

By Proposition5, $\mathrm{Ext}^i_{A^e}(A, A) \cong \mathrm{Ext}^i_{B^e}(B, A)$. By (1) above, $\mathrm{Ext}^i_{B^e}(B, A) \cong \mathrm{Ext}^i_{B^e}(A, A)$. Hence (3) holds. □

Theorem 7. *Let B be an algebra with a stratifying ideal BeB. We denote by A the factor algebra B/BeB. There exist the following two long exact sequences:*

(1) $\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \to \cdots$;
(2) $\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{HH}^n(B) \to \mathrm{HH}^n(A) \oplus \mathrm{HH}^n(eBe) \to \cdots$.

Proof. By Lemma2 and Proposition6 (1), we have the following two long exact sequence:

$$\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{B^e}(A, A) \to \cdots;$$

and

$$\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{B^e}(A, A) \oplus \mathrm{Ext}^n_{B^e}(BeB, BeB) \to \cdots.$$

By Proposition6 (2) and (3), we have the following two long exact sequences:

$$\cdots \to \mathrm{Ext}^n_{B^e}(B, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{A^e}(A, A) \to \cdots ;$$

and

$$\cdots \to \mathrm{Ext}^n_{B^e}(A, BeB) \to \mathrm{Ext}^n_{B^e}(B, B) \to \mathrm{Ext}^n_{A^e}(A, A) \oplus \mathrm{Ext}^n_{eBe^e}(eBe, eBe) \to \cdots .$$

□

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