QF RINGS AND QF ASSOCIATED GRADED RINGS

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ABSTRACT. The associated graded ring of QF (quasi-Frobenius, generally not commutative) ring R is not necessarily QF. We shall prove that the associated graded ring of R is QF if and only if R is QF and for any primitive idempotent e the upper Loewy series of Re and eR is coincident with the lower Loewy series of Re and eR respectively.

In connection with the above result we consider for any pair of positive integers t, n a ring $\Lambda = K[x_0, x_1, \cdots, x_n]/(x_i^t - \frac{1}{x_i}\prod_{j=0}^{n} x_j \mid i = 0, 1, \cdots, n)$, because for $t \neq n$, the associated graded ring of Λ is different from Λ but they are both QF (=0-dimensional Gorenstein). So we expect that for $t = n$, Λ is Gorenstein even if Krull dimension > 0. We pointed out however that if $t = n = 2$, Λ is not Gorenstein, but Cohen-Macauley. Further if $n = t = 3$, Λ is neither Cohen-Macauley nor toric, of course not Gorenstein.

1. A CHARACTERIZATION OF QF ASSOCIATED GRADED RINGS

For an Artinian ring R having the Jacobson radical J with $J^{n+1} = 0$, the series : $R \supset J \supset J^2 \supset \cdots \supset J^n \supset J^{n+1} = 0$ is called the upper Loewy series of ${}_RR$ (resp. R_R). If we put $A_i = J^i/J^{i+1}$, we can naturally define the multiplication of elements $a + J^{i+1} \in A_i$ and $b + J^{j+1} \in A_j$ to be $ab + J^{i+j+1} \in A_{i+j}$. Then by using this multiplication we make the (formal) direct sum $A_0 \oplus A_1 \oplus \cdots \oplus A_n$ into a ring R_G. Clearly this ring R_G is positive $Z-$graded and A_1 generates the radical of R_G. R_G is called the associated graded ring of R. Cf.[3]. R and R_G may be not isomorphic to each other. Cf. Example 2.1

By Morita equivalence [8] we can assume without loss of generality that rings are basic. Let e be a primitive idempotent of ring R. Then $e + J \in A_0$ is a primitive idemotent of R_G which we shall denote by e_G for short. If we denote the right (resp. left) annihilator of a subset M of R by $r(M)$ (resp. $l(M)$), then $Soc\,(Re) = r(J)e$ (resp. $Soc\,(eR) = e\,l(J)$). At first we have

Proposition 1.1. *If ${}_{R_G}Soc(R_Ge_G)$ is simple for a primitive idempotent e_G, then the ${}_RSoc(Re)$ is simple. And if $Soc(R_Ge_G) \simeq R_Gf_G/Rad(R_G)f_G$ for a primitive idempotent f, then $Soc(Re) \simeq Rf/Jf$.*

Proof. Let $J^\rho e \neq 0$ and $J^{\rho+1}e = 0$. Then $A_\rho e_G \neq 0$. Let us denote the set $\{\alpha \in R_G | A_1\alpha = 0\}$ by $r(A_1)$. Since A_1 generates the radical of R_G, by the assumption $Soc(R_Ge_G) = r(Rad(R_G))e_G = r(A_1)e_G$ is a unique simple R_G-submodule of R_Ge_G. Hence $r(A_1)e_G \subseteq A_\rho e_G$. On the other hand $r(A_1)e_G \supseteq A_\rho e_G$ by $A_1A_\rho e_G = 0$. Hence $r(A_1)e_G = A_\rho e_G$.

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Now take $u \in r(J)e$. Then there is a unique positive integer j such that $u \in J^j e \setminus J^{j+1}e$. As $au = 0$ for any $a \in J$ we have that $(a + J^2)(u + J^{j+1}) = au + J^{(j+1)+1} = 0 + J^{(j+1)+1} \in A_{j+1}$ for $a + J^2 \in A_1$. Therefore $u + J^{j+1} \in r(A_1)e_G = A_\rho e_G$ and it follows that $j = \rho$. This implies that $r(J)e \subseteq J^\rho e$. On the other hand $r(J)e \supseteq J^\rho e$ by $J(J^\rho e) = 0$. Hence $J^\rho e = r(J)e$. Further $J^\rho e$ can be identified with $A_\rho e_G$ because $J^{\rho+1}e = 0$.

Now R/J can be identified with $R_G/Rad(R_G)$. From $Rad(R_G)^\rho e_G = A_\rho e_G$ is simple as a left R_G-module it follows that $r(J)e$ is a simple R-module.

The latter statement follows from that $f_G A_\rho e_G \neq 0$ if and only if $f J^\rho e \neq 0$. This completes the proof. □

Following Thrall [12] a ring R is said to be left QF-2 if the socle of Re is simple for every primitive idempotent e. Then we have immediately

Corollary 1.2. *R is left QF-2 if R_G is left QF-2.*

Now we shall prove

Theorem 1.3. *If R_G is QF, then R is QF.*

Proof. By the assumption that R is basic there is a set of primitive idempotents e_i such that $1 = \sum_{i=1}^{n} e_i$ and $Re_i \not\simeq Re_j$ for $i \neq j$.

Since R_G is QF, for all e_i, $Soc(R_G e_{i_G}) = r(Rad(R_G))e_{i_G} = r(A_1)e_{i_G}$ (resp. $Soc(e_{i_G} R_G) = e_{i_G} l(Rad(R_G)) = e_{i_G} l(A_1)$) is a simple left (resp. right) R_G -module.

Hence by Proposition 1.1 $r(J)e_i$ (resp. $e_i l(J)$) is a simple left (resp. right) R -module.

On the other hand it holds that ${}_R l(J)e_i \simeq {}_R Hom_R((e_i R/e_i J)_R, {}_R R_R)$ and as is quoted above ${}_R l(J)e_i$ is simple. Similarly $e_i r(J)_R \simeq Hom_R((Re_i/Je_i, {}_R R_R)$ is simple.

Therefore by [6, Theorem 2.1] it holds the duality $Hom_R(-, {}_R R_R)$ between the categories of finitely generated left R-modules and right R-modules. and hence R is QF. Cf. also [5]. □

In Example 2.1 we shall show that both the converses of Corollary 1.2 and Theorm 1.3 do not hold.

Now it needs to give a characterization of QF ring R for which the associated graded ring R_G is QF.

We say that the series : $Re = r(J^{\rho+1})e \supset r(J^\rho)e \supset r(J^{\rho-1})e \supset \cdots \supset r(J)e \supset r(R)e = 0$ is the lower Loewy series of Re.

In their book [2] Artin-Nesbitt-Thrall proved that subquotient modules $J^k e/J^{k+1}e$ and $r(J^{\rho+1-k})e/r(J^{\rho-k})e$ have non-zero isomorphic constituents for every $0 \leqq k \leqq \rho$. Then we have the following question:

Which kind of rings do satisfy the coincidence of every non-zero isomorphic constituent of $J^k e/J^{k+1}$ with $J^k e/J^{k+1}e$ itself ?

We can provide Proposition 1.4 as an answer to the question.

A positive Z-graded ring $R = A_0 \oplus A_1 \oplus \cdots \oplus A_n$ is called to be standard if A_1 generates the radical of R.

Then we have

Proposition 1.4. *If R is a standard positive Z-graded QF ring , then the upper Loewy series of Re coincides with the lower Loewy series of Re for any primitive idempotent e.*

Proof. For a primitive idempotent e let $Re = A_0e \oplus A_1e \oplus A_2e \oplus \cdots \oplus A_\rho e, \quad \rho \leqq n$, be a grading of Re. Then $A_iA_je \subseteq A_{i+j}e$ and the rad(Re)= $A_1e \oplus A_2e \oplus \cdots \oplus A_\rho e$.

Then from the assumption that R is QF it holds that ${}_Rl(J)e = r(J)e = Soc(Re) = J^\rho e = A_\rho e$.

Assume that $r(J^s)e = J^{\rho+1-s}e$ for an integer $s \geq 1$ (as pointed out above this assumption is satisfied for $s = 1$), and suppose that $r(J^{s+1})e \neq J^{\rho-s}e$ for $r(J^{s+1})e \supseteq J^{\rho-s}e$.

Then there is $0 \neq y = \sum_{l \leqq j < \rho - s} y_j \in r(J^{s+1})e$ such that $0 \neq y_j \in A_je$. From $0 = J^{s+1}y = J^s(Jy)$ it follows $Jy \in r(J^s)e = J^{\rho+1-s}e = \oplus_{\rho+1-s \leqq k} A_ke$.

On the other hand $Jy = \sum_{j < \rho - s} Jy_j \in \oplus_{l \leqq j < \rho - s} A_{j+1}e = \oplus_{l+1 \leqq k < \rho+1-s} A_ke$.

Hence $Jy = 0$. Then $A_1y_l = 0$ since A_1 generates J and $y_j \in A_je$ for $l \leqq j < \rho - s$. This implies $y_l \in A_\rho e$ and this is a contradiction because $l \neq \rho$.

Therefore we conclude that $r(J^{s+1})e = J^{\rho-s}e$.

Now by induction on s we complete the proof. □

Corollary 1.5. *If R_G is QF then for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively.*

Proof. Let $J^\rho e \neq 0$ but $J^{\rho+1}e = 0$. Then by Proposition 1.4 it follows that
$(Rad(R_G))^{\rho+1-k}e_G = Soc^k(R_Ge_G) = r((Rad(R_G))^k)e_G$ for $k = 1, 2, \cdots, \rho$.

Now we want to prove that $J^{\rho+1-k}e = Soc^k(Re) = r(J^k)e$ for $k = 1, 2, \cdots, \rho$.

Suppose $x \in r(J^k)e \setminus J^{\rho+1-k}e$ since $J^{\rho+1-k}e \subseteq r(J^k)e$. Let j be the maximal integer such that $x \in J^j \setminus J^{j+1}e$. Then $j < \rho + 1 - k$. For $x + J^{j+1} \in A_je_G$ it holds that $A_k(x + J^{j+1}e) = (J^k/J^{k+1})(x + J^{j+1}e = 0 + J^{j+1+k}e = 0$. This implies that $x + J^{j+1}e \in r(A_k)e_G = r(A_1^k)e_G = r((Rad(R_G))^k)e_G = Rad(R_G)^{\rho+1-k}e_G = (A_{\rho+1-k} \oplus A_{\rho+2-k} \oplus \cdots)e_G$. Thus we have $j \geq \rho + 1 - k$. But this contradicts to $j < \rho + 1 - k$.

We can prove similarly that $eJ^{\sigma+1-k} = Soc^k(eR) = e\,l(J^k)$ for $k = 1, 2, \cdots, \sigma$, where $eJ^\sigma \neq 0$ but $eJ^{\sigma+1} = 0$. This completes the proof. □

Proposition 1.6. *If R is QF and for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively, then the associated graded ring R_G is QF.*

Proof. Let $J^\rho e \neq 0$ but $J^{\rho+1}e = 0$. From the coincidence of series of the upper Loewy series and the lower Loewy series of Re it follows that $r(J^i)e = Soc^i({}_RRe) = J^{\rho+1-i}e$, $i = 1, 2, \cdots, \rho$. And especially $Soc(Re) = J^\rho e$ is a simple left R-module since R is QF.

For $x \in J^k e \setminus J^{k+1}e$ and $k \leqq \rho - 1$ suppose $Jx \in J^{k+2}e = Soc^{\{\rho+1-(k+2)\}}(Re)$. Then $J^{\{\rho+2-(k+2)\}}x = 0$ and $x \in r(J^{\rho-k})e = J^{k+1}e$. But this is a contradiction.

Therefore if $k \leqq \rho - 1$ and if $x + J^{k+1}e \neq 0 \in A_k e_G = (J^k/J^{k+1})e_G$ it holds that $0 \neq A_1(x + J^{k+1}e) \in A_{k+1}e_G$. Therefore $Soc(R_G e_G) = r(Rad(R_G))e_G \subseteq J^\rho e = Rad(R_G)^\rho e_G$.

As $r(Rad(R_G))e_G \supseteq Rad(R_G)^\rho e_G$ we have $Soc(R_G e_G) = r(Rad(R_G))e_G = J^\rho e$, which can be considered as a simple left R_G-module because $J^\rho e$ is a simple left R-module.

Now it is clear that $Hom_{R_G}(e_G R_G/e_G Rad(R_G), {}_{R_G}R_{G R_G}) \simeq r(Rad(R_G))e_G$.
This implies that the dual module $Hom_{R_G}(e_G R_G/e_G Rad(R_G), {}_{R_G}R_{G R_G})$ of a simple right R_G-module $e_G R_G/e_G Rad(R_G)$ is a simple left R_G-module.

Similarly we have that the dual module $Hom_{R_G}(R_G e_G/Rad(R_G)e_G, {}_{R_G}R_{G R_G})$ of a simple left R_G-module $R_G e_G/Rad(R_G)e_G$ is a simple right R_G-module.

Therefore by [6, Theorem 2.1] it holds the duality $Hom_{R_G}(-, {}_{R_G}R_{G R_G})$ between the categories of finitely generated left R_G-modules and finitely generated right R_G-modules. Hence R_G is QF. $\square$

Now by Propositions 1.5 and 1.6 we have the following characterization of QF associated graded rings:

Theorem 1.7. *The following conditions $(i), (ii)$ and (iii) are equivalent to each other:*

(i) The associated graded ring R_G is QF,

(ii) R is QF and for any primitive idempotent e the upper Loewy series of Re and eR are coincident with the lower Loewy series of Re and eR respectively,

(iii) R is QF and for any primitive idempotent e_i and integer $0 \leqq k \leqq \rho_i$ it holds that ${}_R J^k e_i/J^{k+1}e_i \simeq {}_R Hom_R(e_i J^{\rho-k}/e_i J^{\rho-k+1}{}_R, {}_R R_R)$ (resp. $e_i J^k/e_i J^{k+1} \simeq$ $Hom_R({}_R J^{\sigma-k}e_i/J^{\sigma-k+1}e_i, {}_R R_R)$, where $J^{\rho_i}e_i \neq 0$ but $J^{\rho_i+1}e_i = 0$ (resp. $e_i J^{\sigma_i} \neq 0$ but $e_i J^{\sigma_i+1} =$ 0).

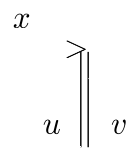
Let π be a Nakayama permutation of QF ring R on the set of all non isomorphic primitive idempotents e_i, $i = 1, 2, \cdots, n$.

Then it holds that ${}_R Re_{\pi(j)}/Je_{\pi(j)} \simeq {}_R Hom_R(e_j R/e_j J_R, {}_R R_R)$.

Corollary 1.8. *R_G is QF if and only if R is QF and for any primitive idempotent e_i it holds that ${}_R J^k e_i/J^{k+1}e_i \simeq \oplus_j^n n_{i,j} \times Re_{\pi(j)}/Je_{\pi(j)}$ for a direct sum decomposition : $e_i J^{\rho-k}/e_i J^{\rho-k+1}{}_R \simeq \oplus_j^n n_{i,j} \times e_j R/e_j J$, where $n_{i,j} \times e_j R/e_j J$ means the direct sum of $n_{i,j}$ copies of $e_j R/e_j J$.*

As indecomposable commutative algebras are local, for them there are no difference between QF-2, QF-3 and QF rings. Hence Theorem 1.7 and Corollary 1.8 are considered to be results for non commutative rings, though Theorem 1.7 seems to be a generalization of Iarrobino's result [4; Proposition 1.7] for 0-dimensional Gorenstein algebras.

Example 2.1. (i) Let R be an algebra over a field K defined by the following quiver.



$$x^3 = vu,\quad 0 = uv,\quad 0 = xv,\quad 0 = ux.$$

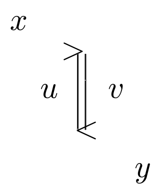
Then the K-bases : $R = \{e_1, x, x^2, x^3, u; e_2, v\}$, $J = Rad(R) = \{x, x^2, x^3, u; v\}$, $J^2 = \{x^2, x^3\}$,$J^3 = \{x^3\}$ and $J^4 = \{0\}$. By $0 \neq x^3 = vu$ and $0 = uv$, R is not commutative. As $r(J)e_1 = Kx^3 \simeq Re_1/Je_1$ and $r(J)e_2 = Kv \simeq Re_1/Je_1$, R is left QF-2.

It happens however that $(v + J^2)(u + J^2) = vu + J^3 = 0 \in G(R)$ for the contrary $vu = x^3 \neq 0 \in R$. Then $r(rad(G(R)))e_1 = K(u + J^2) + K(x^3 + J^4) \simeq G(R)e_2/Rad(G(R))e_2 \oplus G(R)e_1/Rad(G(R))e_1$ is not simple. Hence $G(R)$ is not left QF-2.

This shows that the converse of Proposition 1.1 does not hold.

The next example (ii) shows that the converse of Theorem 1.3 does not hold.

(ii)



$$x^3 = vu,\quad y^2 = uv,\quad 0 = xv,\quad 0 = ux,\quad 0 = vy,\quad 0 = yu.$$

Then the K-bases $= \{e_1, x, x^2, x^3, u, e_2, y, v\}$, where e_2 and e_2 are primitive idempotents.

By $x^3 = vu$ and $y^2 = uv$, R is not commutative. $J = Rad(R) = \{x, x^2, x^3, u; y, y^2, v\}$, $J^2 = \{x^2, x^3, y^2\}$,$J^3 = \{x^3\}$, $J^4 = 0$. $r(J) = \{x^3, y^2\} = l(J), r(J)e_1 = Kx^3 \simeq Re_1/Je_1$ and $r(J)e_2 = Ky^2 \simeq Re_2/Je_2$. Hence R is QF.

As $(v+J^2)(u+J^2) = 0+J^3$ and $(y+J^2)(u+J^2) = 0+J^3, Soc(R_G(e_1)_G) = K(u+J^2)+K(x^3 + J^3) \simeq R_G(e_2)_G/Rad(R_G)(e_2)_G \oplus R_G(e_1)/Rad(R_G)(e_1)_G)$. Hence $Soc(R_G(e_1)_G)$ is not simple. Therefore R_G is not QF.

We know that the upper Loewy series of Re_1 and Re_2 are $(1, 1+2, 1, 1)$ and $(2, 1+2, 2)$ respectively. On the other hand lower Loewy series of Re_1 and Re_2 are $(1, 1, 2+1, 1)$ and $(2, 1+2, 2)$ respetively. From Theorem 1.7 it follows also that R_G is not QF.

Example 2.2. Let Λ be a quotient ring $K[x_0, x_1, \cdots, x_n]/I$ such that the ideal I are generated by $n+1$ polynomials $x_i^t - \dfrac{1}{x_i}\displaystyle\prod_{j=0}^{n} x_j$, $i = 0, 1, \cdots, n$, for the pairs (t, n).

In case of $t \neq n$, for $min\{n,t\} \leqq |t-n|s < max\{n,t\}$ there is an idempotent $e \equiv \prod_{i=0}^{n} x_i^{|t-n|s}$ mod I of Λ and $\Gamma = (1-e)\Lambda \simeq K[x_0, x_1, \cdots, x_n]/(\prod_{i=0}^{n} x_i^{|t-n|s}, I)$ is an Artinian local algebra. Cf.[12] and Kikumasa-Yoshimura [6].

Let us denote the associated graded algebra $\Gamma_G = A_0 \oplus A_1 \cdots \oplus A_m$. Then if $t > n$, $m = (t+1)(n-1)$ and $dim_K(A_k) = \sharp\{(d_0, d_1, \cdots, d_i, \cdots d_j, \cdots, d_n) | \sum_{l=0}^{n} d_l = k, 0 \leqq d_l \leqq t+1$ and $d_i = d_j = 0$ for $i \neq j\}$. Hence $dim_K A_k = dim_K A_{m-k}$. It follows by Corollary 1.8 that Γ_G (and hence by Theorem 1.3 Γ) is QF.

If $t < n$, $m = (n+1)(t-1)$ and $dim_K(A_k) = \sharp\{(d_0, d_1, \cdots, d_n) | \sum_{l=0}^{n} d_l = k, 0 \leqq d_l \leqq t-1\}$. Hence we have similarly $dim_K A_k = dim_K A_{m-k}$ and Γ_G (and hence Γ) is QF.

Now we can extend our consideration for Λ to the case of $n = t$. Then as $\frac{1}{x_i}\prod_{j=0}^{n} x_j \equiv x_i^n$ mod I for $i = 1, 2, \cdots, n$, Λ is a positive Z-graded with respect to homogeneous elements $\bar{x}_0, \bar{x}_1, \cdots, \bar{x}_n$ of degree 1. If we put $u = \prod_{i=1}^{n} \bar{x}_i$, then $K[u]$ is a subalgebra of Λ, which is a polynomial ring over K of a variable u . Further all $\bar{x}_i$'s satisfy the equation $X^{n+1} - u = 0 \in K[u][X]$. Hence by Noether's normalization theorem the Krull dimension of $\Lambda = 1$.

Let $t = n = 1$, then the generators of I are formally $\{x_0 - \frac{1}{x_0}x_0x_1 = x_0 - x_1, x_1 - \frac{1}{x_1}x_0x_1 = x_1 - x_0\} = \{x_0 - x_1\}$ and Λ is a polynomial ring of one variable and is obviously Gorenstein.

Let $t = n = 2$. Then $\{f_2 = x_0x_1 - x_2^2, f_0 = x_1x_2 - x_0^2, f_1 = x_2x_0 - x_1^2\}$ defines an intersection of quadratic cones. In [9] Stanley commented that the following Theorm 2.1 was proved first by Macauley.

Theorem 2.1. *If a K-algebra Λ is standard positively Z-graded and Gorenstein of Krull dimension d, then for Poincaré series $F(\Lambda, \lambda)$ it holds that $F(\Lambda, \frac{1}{\lambda}) = (-1)^d \lambda^\rho F(\Lambda, \lambda)$ (as rational functions of λ) for some integer ρ.*

By the Buchberger's algorithm we obtain the reduced Gröbner bases $\{f_0, f_1, f_2, f_3 = S(f_0, f_1) = -x_0^3 - x_1^3\}$ of I with respect to the degree- lexicographical order $x_0 < x_1 < x_2$.

As the leading terms are $Lt(f_0) = x_0^0x_1^1x_2^1, Lt(f_1) = x_0^1x_1^0x_2^1, Lt(f_2) = x_0^0x_1^0x_2^2, Lt(f_3) = x_0^0x_1^3x_2^0$ it holds $\alpha_1 < 3, \alpha_2 < 2$ for the standard bases $\bar{x}_0^{\alpha_0}\bar{x}_1^{\alpha_1}\bar{x}_2^{\alpha_2} \in \Lambda = K[x_0, x_1, x_2]/I$.

Therefore we know that

$$\{\bar{1}, \bar{x}_0, \bar{x}_0^2, \bar{x}_0^3, \cdots, \bar{x}_1, \bar{x}_0\bar{x}_1, \bar{x}_0^2\bar{x}_1, \bar{x}_0^3\bar{x}_1, \cdots, \bar{x}_1^2, \bar{x}_0\bar{x}_1^2, \bar{x}_0^2\bar{x}_1^2, \bar{x}_0^3\bar{x}_1^2, \cdots, \bar{x}_2\}$$

are the K-bases of Λ. Cf. [1:Theorem 1.7.4 and Proposition 2.1.6].

$A_0 = K\,\bar{1}$, $A_1 = K\,\bar{x}_0 + K\,\bar{x}_1 + K\,\bar{x}_2$, $A_2 = K\,\bar{x}_0^2 + K\,\bar{x}_1\bar{x}_0 + K\,\bar{x}_1^2$,

$A_n = K\,\bar{x}_0^n + K\,\bar{x}_1\bar{x}_0^{n-1} + K\,\bar{x}_1^2\bar{x}_0^{n-2}$ for $n \geq 3$,

are $Z^+ \cup \{0\}$-grading of Λ and the set of homogeneous generators is $\{\bar{x}_0, \bar{x}_1, \bar{x}_2\}$ with degree 1.

Thus the Poincarė series $F(\Lambda, \lambda) = 1 + \sum_{n=1}^{\infty} 3\lambda^n = \frac{3}{1-\lambda} - 2 = \frac{2\lambda+1}{1-\lambda}$.

Now there is no ρ which satisfies $(-1)^1 \lambda^\rho F(\Lambda, \lambda) = (-1)\lambda^\rho \frac{2\lambda+1}{(1-\lambda)} =$

$= \frac{\lambda+2}{(\lambda-1)} = \frac{2\frac{1}{\lambda}+1}{1-\frac{1}{\lambda}} = F(\Lambda, \frac{1}{\lambda})$. Therefore by Theorem 2.1 Λ is not Gorenstein.

By the way we notice that in this case Λ is Cohen-Macaulay because $\Lambda = K[\bar{x}_0] \oplus K[\bar{x}_0]\ \bar{x}_1 \oplus K[\bar{x}_0]\ \bar{x}_2$ is $K[\bar{x}_0]$-free module. Here we notice that $K[\bar{x}_0]\ \bar{x}_1^2 \subset K[\bar{x}_0]\ \bar{x}_2$ by $x_0 x_2 \equiv x_1^2 \mod I$.

This arises a new question whether Λ is Cohen-Macaulay.

In order to answer the question let us consider Λ for $n = t = 3$. In this case the binomials $\{f_3 = x_0x_1x_2 - x_3^3, f_0 = x_1x_2x_3 - x_0^3, f_1 = x_2x_3x_0 - x_1^3, f_2 = x_3x_0x_1 - x_2^3\}$ generates I and by using the Buchberger's algorithm we obtain the following Gröbner bases

$Gr = \{f_0, f_1, f_2, f_3, f_4 = S(f_0, f_1) = x_0^4 - x_1^4, f_5 = S(f_1, f_2) = x_1^4 - x_2^4,$
$f_6 = S(f_0, f_3) = x_0x_1^2x_2^2 - x_0^3x_3^2, f_7 = S(f_1, f_3) = x_0^2x_1x_2^2 - x_1^3x_3^2,$
$f_8 = S(f_2, f_3) = x_0^2x_1^2x_2 - x_2^3x_3^2, f_9 = S(f_2, S(f_0, f_1)) = x_1^3x_2^3 - x_0^5x_3\}$ of I

and the leading terms $\{Lt(f_0) = x_1x_2x_3, Lt(f_1) = x_2x_3x_0, Lt(f_2) = x_3x_0x_1, Lt(f_3) = x_3^3,$ $Lt(f_4) = x_1^4, Lt(f_5) = x_2^4, Lt(f_6) = x_0^3x_3^2, Lt(f_7) = x_1^3x_3^2, Lt(f_8) = x_2^3x_3^2, Lt(f_9) = x_0^5x_3\}$ with respect to the degree-lexicographical order $x_0 < x_1 < x_2 < x_3$.

Now there is no positive integer n such that $x_0^n \xrightarrow{Gr}_+ 0$. Therefore $K[\bar{x}_0]$ is a polynomial ring in the variable $\bar{x}_0$. Further $f_6 = x_0x_1^2x_2^2 - x_0^3x_3^2 = x_0(x_1^2x_2^2 - x_0^2x_3^2) \in I$ and $(x_1^2x_2^2 - x_0^2x_3^2) \notin I$ because the terms $x_1^2x_2^2$ and $x_0^2x_3^2$ are not reduced by any Gröbner base. Hence $\bar{x}_0\ (\bar{x}_1^2\bar{x}_2^2 - \bar{x}_0^2\bar{x}_3^2) = \bar{0}$, but $(\bar{x}_1^2\bar{x}_2^2 - \bar{x}_0^2\bar{x}_3^2) \neq \bar{0}$.

Hence Λ is not a $K[\bar{x}_0]$-free module. This implies Λ is not Cohen-Macaulay.

As all generators of I are binomials, Λ may be a toric variety. Cf. [10].

Toric varieties are defined to be Noetherian integral domains. However as we prove just now Λ has a zero divisor we cannot expect that Λ is toric.

Proposition 2.2. *If $n = t = 3$, then Λ is neither Cohen-Macaulay nor toric. Of course Λ is not Gorenstein.*

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