BROUÉ'S ABELIAN DEFECT GROUP CONJECTURE

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First of all, the result which will be presented here right now is actually a joint work with **Naoko Kunugi** and **Katsushi Waki** [9], which should show up as an ordinary paper in near future, hopefully. In representation theory of finite groups, especially in modular representation theory of finite groups there are quite a few problems and conjectures which are pretty much interesting and important. I believe that most of them have origins which are (were) due to Richard Brauer (1901–77) who should have been a unique pioneer of modular representation theory of finite groups. However, I guess most people might agree that the following conjectures should be the ones if we have to choose three of them from the problems and conjectures. Namely, *Alperin's weight conjecture* (1986), *Dade's conjecture* (1990) and *Broué's abelian defect group conjecture* (Broué's ADGC) (1988). In this short note we shall focus on Broué's ADGC, particularly.

Michel Broué around late eighties announced the following conjecture, which is nowadays well-known and is called Broué's ADGC (abelian defect group conjecture). That is,

Broué's abelian defect group conjecture (ADGC) (see [2], [3], [4], [5])**.** Let p be a prime and let G be a finite group. Let a triple $(\mathcal{O}, \mathcal{K}, k)$ be a p-modular system, namely, $\mathcal{O}$ is a complete discrete valuation ring of rank one, $\mathcal{K}$ is the quotient field of $\mathcal{O}$ with characteristic zero, and k is the residue field $\mathcal{O}/\mathrm{rad}(\mathcal{O})$ with characteristic p. We assume moreover that the p-modular system $(\mathcal{O}, \mathcal{K}, k)$ is big enough for all subgroups of G, namely, $\mathcal{K}$ and k are both splitting fields for all subgroups of G. Now, let A be a block algebra of the group algebra $\mathcal{O}G$ with defect group P, and set $H = N_G(P)$, the normalizer of P in G. Let B be a block algebra of $\mathcal{O}H$ which is the Brauer correspondent of A, and hence B has the same defect group P. If the defect group P of A is abelian (commutative), then the algebras A and B should be derived (Rickard) equivalent, namely,

$$\mathrm{D}^b(\text{mod-}A) \simeq \mathrm{D}^b(\text{mod-}B) \quad \text{(equivalent)}$$

as triangulated categories, where mod-A is the category of finitely generated right A-modules, and $\mathrm{D}^b(\mathfrak{A})$ is the bounded derived category of an abelian category $\mathfrak{A}$.

The detailed version of this paper will be submitted for publication elsewhere.

As well-known there is a wonderful and beautiful result due to Jeremy Rickard ([12], [13]), which characterizes such a derived equivalence completely and that is a generalization of a Morita equivalence from modules to complexes of modules. In fact, in the conjecture above, a stronger conclusion is expected. That is, *derived (Rickard) equivalent* could be replaced by *splendidly (Rickard) equivalent*, which is due to Jeremy Rickard [13]. As far as we know, so far there have never existed any counter-example even to the stronger version of Broué's ADGC (or we might want to call it Rickard's version of Broué's ADGC).

There are several results, where Broué's ADGC has been checkd in particular cases. We do not want to mention them completely in detail, however, we want to say a few words on a specific case. Namely, for the case where our abelian defect group P is just $C_3 \times C_3$, the elementary abelian group of order 9. The auther with Naoko Kunugi finally have proved that Broué's ADGC is true for the case where our block algebra A is the principal block algebra and our defect group of A is $P = C_3 \times C_3$ (and it turns out that P is a Sylow 3-subgroup of G), see [6]. However, we should confess that we used a lot of iniciated wonderful and important works done by L. Puig, T. Okuyama, H. Miyachi, ... and also the classification of finite simple groups (which we do not like, to be honest, as a matter of fact, but we had no the other choice, life is tough ...). And then, we were successful to check that Broué's ADGC is true also for non-principal block algebras A with the same defect group $P = C_3 \times C_3$ for specific sporadic simple groups G, see [7] and [8].

Anyhow, our work presented here is a sort of continuation of this project. Our main result is the following:

Theorem (Koshitani-Kunugi-Waki, 2005 [9])**.** *Keep the notation as in the conjecture Broué's ADGC above. Let $p = 3$, and let G be the Janko simple group J_4. Let A be a unique non-principal block algebra of $\mathcal{O}G$ with defect group $P = C_3 \times C_3$. Then, there exists a splendid Rickard equivalence between A and its Brauer correspondent block algebra B in $\mathcal{O}N_G(P)$. This means that Broué's ADGC holds for J_4 at least for the particilar block A, and it turns out that Broué's ADGC holds for all primes p and for all p-block algebras of J_4.*

Remark. To prove our main result, results of Okuyama in [10] and [11] play imporant rôles. In order to know that Broué's ADGC holds for all primes p and for all p-blocks of J_4 after we check it for our particular one single non-principal 3-block of J_4, we need a lemma which is stated in [1].

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References

[1] J. An, E.A. O'Brien, R.A. Wilson, The Alperin's weight conjecture and Dade's conjecture for the simple group J_4, LMS J. Comput. Math. **6** (2003), 119–140.

[2] M. Broué, Isométries de caractères et équivalences de Morita ou dérivées, Publ. Math. IHES **71** (1990), 45–63.

[3] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque **181-182** (1990), 61–92.

[4] M. Broué, Equivalences of blocks of group algebras, in ;Finite Dimensional Algebras and Related Topics, (edited by V. Dlab and L.L. Scott) Kluwer Acad. Pub., Dordrecht, 1994, pp.1–26.

[5] S. König, A. Zimmermann, Derived Equivalences for Group Rings, in: Lecture Notes in Math., Vol.**1685**, Springer, Berlin, 1998.

[6] S. Koshitani, N. Kunugi, Broué's conjecture holds for principal 3-blocks with elementary abelian defect group of order 9, J. Algebra **248** (2002), 575–604.

[7] S. Koshitani, N. Kunugi, K. Waki, Broué's conjecture for non-principal 3-blocks of finite groups, J. Pure Appl. Algebra **173** (2002), 177–211.

[8] S. Koshitani, N. Kunugi, K. Waki, Broué's abelian defect group conjecture for the Held group and the sporadic Suzuki group, J. Algebra **279** (2004), 638–666.

[9] S. Koshitani, N. Kunugi, K. Waki, Broué's abelian defect group conjecture holds for the Janko simple group J_4, 2005, in preparation.

[10] T. Okuyama, Some examples of derived equivalent blocks of finite groups, preprint (1997).

[11] T. Okuyama, Remarks on splendid tilting complexes, in: Representation theory of finite groups and related topics, edited by S. Koshitani, RIMS Kokyuroku **1149**, Proc. Research Institute for Mathematical Sciences, Kyoto University, 2000, 53–59.

[12] J. Rickard, Derived equivalences as derived functors, J. London Math. Soc. **(2) 43** (1991), 37–48.

[13] J. Rickard, Splendid equivalences: derived categories and permutation modules, Proc. London Math. Soc. **(3) 72** (1996), 331–358.

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