LINKAGE AND DUALITY OF MODULES

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ABSTRACT. Martsinkovsky and Strooker [5] recently introduced module theoretic linkage using syzygy and transpose. This generalization brings possibility of much application of linkage, especially, to homological theory of modules. In the present paper, we connect linkage of modules to certain duality of modules. We deal with invariance of Gorenstein dimension, characterization of Cohen-Macaulay modules over a Gorenstein local ring using linkage and their generalization to non-commutative algebras.

Let Λ be a left and right Noetherian ring. Let modΛ (respectively, modΛ^{op}) be the category of all finitely generated left (respectively, right) Λ-modules. Throughout the paper, all modules are finitely generated and left modules (if the ring is non-commutatie) and right modules are considered as Λ^{op}-modules. We denote the stable category by $\underline{\mathrm{mod}}\Lambda$, the syzygy functor by $\Omega : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda$, and the transpose functor by $\mathrm{Tr} : \underline{\mathrm{mod}}\Lambda \to \underline{\mathrm{mod}}\Lambda^{\mathrm{op}}$.

Let $T_k := \mathrm{Tr}\Omega^{k-1}$ for $k > 0$ and $\lambda := \Omega\mathrm{Tr}$. Following [5], we define

DEFINITION. A finitely generated Λ-module M and a Λ^{op}-module N are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$, in other words, M is horizontally linked (to λM) if and only if $M \cong \lambda^2 M$.

A rather different definition of linkage of modules is proposed by Yoshino and Isogawa [9]. However, both definitions coincide for Cohen-Macaulay modules over a commutative Gorenstein ring (see [5], section 3).

Let us start with the following simple observation which connect duality with linkage.

PROPOSITION 1. *T_kM is horizontally linked if and only if* $\mathrm{Ext}^k_\Lambda(M,\Lambda) = 0$.

We prepare the facts about Gorenstein dimension from [1]. A Λ-module M is said to have G-dimension zero, denoted by $\text{G-dim}_\Lambda M = 0$, if $M^{**} \cong M$ and $\mathrm{Ext}^k_\Lambda(M,\Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(M^*,\Lambda) = 0$ for $k > 0$. This is equivalent to '$\mathrm{Ext}^k_\Lambda(M,\Lambda) = \mathrm{Ext}^k_{\Lambda^{\mathrm{op}}}(\mathrm{Tr}M,\Lambda) = 0$ for $k > 0$' ([1], Proposition 3.8). For a positive integer k, we say that M has G-dimension less than or equal to k, denoted by $\text{G-dim}_\Lambda M \leq k$, if there exists an exact sequence $0 \to G_k \to \cdots \to G_0 \to M \to 0$ with $\text{G-dim}_\Lambda G_i = 0$ for $(0 \leq i \leq k)$. It follows from [1], Theorem 3.13 that $\text{G-dim}_\Lambda M \leq k$ if and only if $\text{G-dim}_\Lambda \Omega^k M = 0$. If $\text{G-dim}_\Lambda M < \infty$, then $\text{G-dim}_\Lambda M = \sup\{k : \mathrm{Ext}^k_\Lambda(M,\Lambda) \neq 0\}$([1], p. 95).

In the following, the proofs are seen in [6].

Invariance of G-dimension under linkage is studied in [5].

The detailed version of this paper will be submitted for publication elsewhere

THEOREM 2. ([5], Theorem 1) *Let Λ be a semiperfect right and left Noetherian ring and M a Λ-module without projective direct summand. Then the following conditions are equivalent.*

(1) G-dim$_\Lambda M = 0$,

(2) G-dim$_{\Lambda^{\mathrm{op}}} \lambda M = 0$ and M is horizontally linked.

In the rest, we consider a commutative ring case and apply the above results to Cohen-Macaulay modules over a commutative Gorenstein local ring. See [2] for Cohen-Macaulay rings and modules and Gorenstein rings.

Let R be a (commutative) Gorenstein local ring and M a finitely generated R-module. Then there are the following useful equations:

$$(1)\quad \text{G-dim}_R M + \mathrm{depth} M = \dim R$$

$$(2)\quad \mathrm{grade}_R M + \dim M = \dim R,$$

where $\mathrm{grade}_R M := \inf\{k \geq 0 : \mathrm{Ext}_R^k(M, R) \neq 0\}$. The first equality is due to [1], Theorems 4.13 and 4.20 and the second one to [7], Lemma 4.8.

The combination of linkage and duality produces the following characterization of a maximal Cohen-Macaulay module which improves [5], Proposition 8.

THEOREM 3. *Let R be a Gorenstein local ring and M a finitely generated R-module without a projective direct summand. Then the following are equivalent*

(1) M is a maximal Cohen-Macaulay module,

(2) $T_k M$ is horizontally linked for $k > 0$,

(3) λM is a maximal Cohen-Macaulay module and M is horizontally linked.

G-dimension is also described using linkage.

PROPOSITION 4.
Let R be a Gorenstein local ring and M a finitely generated module. Then G-dim$_R M \leq k$ if and only if $T_{i+k} M$ is horizontally linked for $i > 0$.

We apply duality theory on a non-commutative Noetherian ring due to Iyama [4] to the category of Cohen-Macaulay modules. Suppose that Λ is a right and left Noetherian ring. Then the functor T_k gives a duality between the categories $\{X \in \underline{\mathrm{mod}}\Lambda : \mathrm{grade}_\Lambda X \geq k\}$ and $\{Y \in \underline{\mathrm{mod}}\Lambda^{\mathrm{op}} : \mathrm{rgrade}_{\Lambda^{\mathrm{op}}} Y \geq k \text{ and } \mathrm{pd}_{\Lambda^{\mathrm{op}}} Y \leq k\}$ [4], 2.1.(1), where we denote a reduced grade of M by $\mathrm{rgrade}_R M := \{k > 0 : \mathrm{Ext}_R^k(M, R) \neq 0\}$ [3]. Returning to our case, we consider a commutative local ring R and a finitely generated R-module M. Then it holds that $\text{G-dim}_R M \geq \mathrm{grade}_R M$, in general. Moreover, if $\text{G-dim}_R M < \infty$, then M is a Cohen-Macaulay module if and only if $\text{G-dim}_R M = \mathrm{grade}_R M$ by the equations (1) and (2). Thus we can apply the above duality to the category of Cohen-Macaulay R-modules.

A finitely generated module M over a Cohen-Macaulay local ring R is called a Cohen-Macaulay module of codimension k, if $\mathrm{depth} M = \dim M = \dim R - k$. Put the full subcategory

$$\mathcal{C}_k := \{M \in \mathrm{mod} R : M \text{ is a Cohen-Macaulay } R\text{-module of codimension } k\}.$$

In order to give a duality, we need a counterpart of the category $\mathcal{C}_k$. Let M be a finitely generated R-module. We put the full subcategory

$\mathcal{C}'_k :=$

$\{N \in \mathrm{mod}R : \mathrm{rgrade}_R N = \mathrm{pd}_R N = k$ and $\lambda^2 N$ is a maximal Cohen-Macaulay module$\}$,

where $\mathrm{pd}_R N$ stands for a projective dimension of N. Then we have

THEOREM 5. *Let R be a Gorenstein local ring. Let $k > 0$. Then the functor T_k gives a duality between full subcategories $\mathcal{C}_k$ and $\mathcal{C}'_k$.*

Using the above duality, we can characterize a Cohen-Macaulay module of codimension $k > 0$.

COROLLARY 6. *Let M be a finitely generated R-module of $\mathrm{grade}_R M = k > 0$. Then the following are equivalent*

(1) M is a Cohen-Macaulay module of codimension k,

(2) $\mathrm{rgrade}_R T_k M = \mathrm{pd}_R T_k M = k$ and $\lambda^2 T_k M$ is a maximal Cohen-Macaulay module.

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