STABLE EQUIVALENCES RELATED WITH SYZYGY FUNCTORS

YOSUKE OHNUKI

Abstract. Let $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence between finite dimensional self-injective algebras over a field. Then Φ preserves triangles in the triangulated category $\underline{\mathrm{mod}}\,\Lambda$ if and only if Φ commutes with syzygy functors. As an application, we study some stable equivalence induced by socle equivalence.

Key Words: Stable equivalence, Nakayama automorphism.

1. Introduction

Throughout this paper K will be a fixed field, and all algebras will be basic finite dimensional self-injective K-algebras without simple algebra summands. By a module we mean a finite dimensional left module unless otherwise stated, and by $\mathrm{mod}\,\Lambda$ we denote the category of finite dimensional left modules over an algebra Λ. In order to distinguish an equivalence between triangulated categories from an equivalence between the additive categories, we say that a functor is a triangle equivalence if it is an equivalence between triangulated categories.

Happel proved in [2] that the stable category of a self-injective algebra is a triangulated category whose translation is the inverse of the syzygy functor. Keller-Vossieck [4] and Rickard [8] proved that the stable category of a self-injective algebra is triangle equivalent to the quotient category of the bounded derived category by its subcategory consisting of perfect complexes. These results give the motivation which develops invariants (of stable equivalence) arisen from a derived equivalence, or properties of triangulated categories. Pogorzały [7] and Xi [9] proved that the Hochschild cohomology and the representation dimension are invariants under a stable equivalence of Morita type, respectively. Our aim also develops an invariant in order to clear up the difference between a stable equivalence of Morita type, a stable equivalence not of Morita type and a stable equivalence induced by a derived equivalence. We shall show that for self-injective algebras Λ, Λ' and an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$, Φ is a triangle functor if and only if Φ commutes with the syzygy functors. As an application, we shall show that the symmetry is an invariant for some stable equivalence [5] induced by a socle equivalence.

2. A homotopy category

We shall prepare some notations related to homotopy categories. Basic notations and definitions are referred to [3] For an abelian category $\mathcal{A}$, we denote by $X^\bullet = (X^n, d_X^n)$ the

The detailed version of this paper will be submitted for publication elsewhere.

(cochain) complex

$$\cdots \longrightarrow X^{n-1} \xrightarrow{d_X^{n-1}} X^n \xrightarrow{d_X^n} X^{n+1} \longrightarrow \cdots$$

with $d_X^n d_X^{n-1} = 0$ for all integers n. The shift functor T of the category of complexes is defined by $T(X^\bullet)^n = X^{n+1}$.

We denote by $\mathrm{K}(\mathcal{A})$ the homotopy category of $\mathcal{A}$, that is, the residue category of the category of complexes by the homotopy relation. We denote by $\mathrm{K}^-(\mathcal{A})$ or $\mathrm{K}^b(\mathcal{A})$ the full subcategory of $\mathrm{K}(\mathcal{A})$ consisting of bounded above complexes or bounded complexes, respectively.

For $* = -$ or b, a homotopy category $\mathrm{K}^*(\mathcal{A})$ is regarded as a triangulated category whose translation functor is the shift functor T, and for any morphism $f^\bullet : X^\bullet \to Y^\bullet$ in $\mathrm{K}^*(\mathcal{A})$ it induces the triangle in $\mathrm{K}^*(\mathcal{A})$

$$X^\bullet \xrightarrow{f^\bullet} Y^\bullet \xrightarrow{\begin{pmatrix} 1_Y \\ 0 \end{pmatrix}} C(f^\bullet) \xrightarrow{\begin{pmatrix} 0 & 1_{TX} \end{pmatrix}} TX^\bullet.$$

Here $C(f^\bullet) := \left(Y^\bullet \oplus TX^\bullet, \begin{pmatrix} d_Y & Tf \\ 0 & d_{TX} \end{pmatrix} \right)$ is the mapping cone. We will consider the case $\mathcal{A} = \mathcal{P}_\Lambda$, where $\mathcal{P}_\Lambda$ is the full subcategory of $\operatorname{mod}\Lambda$ consisting of projective Λ-modules.

3. A stable equivalence which commutes with syzygy functors

Let Λ be a self-injective algebra. For a Λ-module X, we denote by $\iota_X : X \to I_X$ the injective hull of X. The stable category $\underline{\operatorname{mod}}\,\Lambda$ of Λ has the same objects as $\operatorname{mod}\Lambda$, and a morphism from X to Y in $\underline{\operatorname{mod}}\,\Lambda$ is by definition a residue class in $\operatorname{Hom}_\Lambda(X,Y)/\operatorname{proj}_\Lambda(X,Y)$, where $\operatorname{proj}_\Lambda(X,Y)$ is the subspace of $\operatorname{Hom}_\Lambda(X,Y)$ consisting of morphisms which factor through projective Λ-modules. The syzygy functor $\Omega_\Lambda : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$ is a functor naturally defined by the correspondence of an object X to the kernel of the projective cover of X. Note that the syzygy functor induces the stable equivalence functor $\underline{\operatorname{mod}}\,\Lambda \xrightarrow{\sim} \underline{\operatorname{mod}}\,\Lambda$ which is also called the syzygy functor, denoted by Ω_Λ or Ω again.

Happel showed that the stable category of a self-injective algebra is regarded as a triangulated category [2]. In fact, the translation functor of $\underline{\operatorname{mod}}\,\Lambda$ is given by the inverse Ω_Λ^{-1} of Ω_Λ. For each morphism $\underline{f} : X \to Y$ in $\underline{\operatorname{mod}}\,\Lambda$, the standard triangle $X \xrightarrow{\underline{f}} Y \to Z \to X[1]$ is given by the sequence $X \xrightarrow{\underline{f}} Y \xrightarrow{\underline{g}} Z \xrightarrow{\underline{h}} \Omega^{-1}X$ in the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{\iota_X} & I_X & \longrightarrow & \Omega^{-1}X & \longrightarrow & 0 \\
 & & {\scriptstyle f}\downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & Y & \xrightarrow{g} & Z & \xrightarrow[h]{} & \Omega^{-1}X & \longrightarrow & 0.
\end{array}$$

We denote Z by $C(\underline{f})$ and call it the mapping cone of $\underline{f}$ in $\underline{\operatorname{mod}}\,\Lambda$.

We denote by $\mathrm{K}^{-,b}(\mathcal{P}_\Lambda)$ the full subcategory of $\mathrm{K}^{-}(\mathcal{P}_\Lambda)$ consisting of complexes $X^\bullet$ with bounded cohomology i.e., $\mathrm{H}^n(X^\bullet) = 0$ for $n \ll 0$. Keller-Vossieck and Rickard proved the following result.

Proposition 1. [4][8] *For a self-injective algebra Λ, the stable category of Λ is triangle equivalent to the quotient category $\mathrm{K}^{-,b}(\mathcal{P}_\Lambda)/\,\mathrm{K}^b(\mathcal{P}_\Lambda)$.*

Using the similar correspondence on Proposition 1, we show the our main theorem.

Theorem 2. *Assume that there is an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ for self-injective algebras Λ and Λ'. Then the following conditions are equivalent.*

(1) Φ *is a triangle functor.*

(2) Φ *commutes with the syzygy functors i.e., $\Omega_{\Lambda'}\Phi \simeq \Phi\Omega_\Lambda$.*

In [1, Chapter X], Auslander-Reiten-Smalø proved that an equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ commutes with syzygy functors if Λ and Λ' are symmetric algebras. Therefore the following corollary follows from Theorem 2.

Corollary 3. *If Λ and Λ' are symmetric algebras, then any stable equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ is a triangle functor.*

4. Nakayama automorphisms and some application

We recall the definitions and some basic properties of the Nakayama automorphism and the Nakayama functor for a self-injective algebra. Refer to [10] in detail. Let M be a Λ-module and f an automorphism of Λ. The Λ-module ${}_fM$ is the K-space M with the Λ-module structure: $a \cdot m = f(a)m$ for $a \in \Lambda$, $m \in M$. Similarly we define N_f for a right Λ-module N. For a self-injective algebra Λ, there is an automorphism ν of Λ such that Λ and $(D\Lambda)_\nu$ are isomorphic as Λ-bimodules, where $D = \mathrm{Hom}_K(-, K)$. Such an automorphism ν is uniquely determined up to inner automorphisms, and called the Nakayama automorphism of Λ. The Nakayama functor is defined by $\mathcal{N} = D\,\mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda$, and therefore $\mathcal{N}$ is isomorphic to ${}_\nu\Lambda \otimes_\Lambda -$. Note that $\mathcal{N}$ naturally induces the equivalence $\mathrm{mod}(\Lambda/\operatorname{soc}\Lambda) \xrightarrow{\sim} \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)$, and the stable equivalence $\underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda$ of Morita type, which are also denoted by $\mathcal{N}$.

A K-linear map $\lambda : \Lambda \to K$ is associated to a Λ-bimodule isomorphism $\varphi : \Lambda \xrightarrow{\sim} (D\Lambda)_\nu$ if $\lambda = \varphi(1_\Lambda)$. Then λ is non-degenerate and satisfies $\lambda(\nu(x)y) = \lambda(yx)$ for any $x, y \in \Lambda$.

We denote $\overline{a}$ the residue class of $a \in \Lambda$ in $\Lambda/\operatorname{soc}\Lambda$. A Λ-bimodule isomorphism $\Lambda \xrightarrow{\sim} (D\Lambda)_\nu$ induces $\Lambda/\operatorname{soc}\Lambda$-bimodule isomorphism $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$, where $\overline{\nu}$ is an algebra automorphism of $\Lambda/\operatorname{soc}\Lambda$ given by $\overline{\nu}(\overline{a}) = \overline{\nu(a)}$. Therefore we have a K-bilinear map $\overline{\lambda} : \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \times \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \to K, (\overline{a}, \overline{b}) \mapsto \lambda(ab)$ associated with $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$.

Lemma 4. *Let $\overline{\lambda} : \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \times \operatorname{rad}\Lambda/\operatorname{soc}\Lambda \to K$ be a K-bilinear map associated with $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} (D(\operatorname{rad}\Lambda/\operatorname{soc}\Lambda))_{\overline{\nu}}$. Then $\overline{\lambda}(\overline{a}, \operatorname{rad}\Lambda/\operatorname{soc}\Lambda) \neq 0$ for any non-zero $\overline{a}$ in $\operatorname{rad}\Lambda/\operatorname{soc}\Lambda$.*

For a Λ-module homomorphism $f : X \to Y$, we define a Λ-module homomorphism between ${}_\rho X$ and ${}_\rho Y$ for an algebra automorphism ρ of Λ as follows

$$ {}_\rho\Lambda \otimes f : {}_\rho X \to {}_\rho Y, x \mapsto f(x). $$

Lemma 5. *Let A be a (not necessarily self-injective) algebra, and ρ an algebra automorphism of A. Then the following conditions are equivalent.*

(1) *ρ is an inner automorphism.*
(2) *There is an A-module isomorphism $\psi : A \xrightarrow{\sim} {}_\rho A$ such that $\psi f = ({}_\rho A \otimes f)\psi$ for any A-module endomorphism f of A.*

Lemma 5 gives the essential condition in order to characterize symmetry for a self-injective algebra Λ. Therefore we have the following lemma on a notion of module category.

Proposition 6. *The following conditions are equivalent for a self-injective algebra Λ over an algebraically closed field K.*

(1) *Λ is symmetric.*
(2) *The Nakayama functor $\mathcal{N} : \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda) \xrightarrow{\sim} \mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)$ is isomorphic to the identity functor $id_{\mathrm{mod}(\Lambda/\operatorname{soc}\Lambda)}$.*
(3) *$\overline{\nu} : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda/\operatorname{soc}\Lambda$ is an inner automorphism.*

Proposition 6 is not true if K is not algebraically closed field. In [6], we can see the counter-example.

Let $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence for self-injective algebras. We have that $\Phi\tau \simeq \tau'\Phi$, where τ and τ' are stable equivalences induced by Auslander-Reiten translations of Λ and Λ', respectively. By [1], it follows that $\tau \simeq \mathcal{N}\Omega_\Lambda^2$. If Φ is a triangle functor, then we have

$$\Phi\mathcal{N}\Omega_\Lambda^2 \simeq \mathcal{N}'\Omega_{\Lambda'}^2\Phi \simeq \mathcal{N}'\Phi\Omega_\Lambda^2,$$

therefore it follows that $\Phi\mathcal{N} \simeq \mathcal{N}'\Phi$.

In order to preserve the symmetry for triangle stable equivalence $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ between symmetric algebra Λ and self-injective algebra Λ', we consider the problem whether $\Phi \simeq \mathcal{N}'\Phi$, equivalent to $id_{\underline{\mathrm{mod}}\,\Lambda} \simeq \mathcal{N}'$, implies that Λ' is also symmetric. However, this is open in general.

Theorem 7. [5] *Let Λ and Λ' be socle equivalent self-injective algebras, say $p : \Lambda/\operatorname{soc}\Lambda \xrightarrow{\sim} \Lambda'/\operatorname{soc}\Lambda'$. Assume that there are non-degenerate K-linear maps $\lambda : \Lambda \to K$ and $\lambda' : \Lambda' \to K$ such that $\lambda(ab) = \lambda'(a'b')$ for all $a, b \in \operatorname{rad}\Lambda$ and $a', b' \in \operatorname{rad}\Lambda'$ with $\bar{a}' = p(\bar{a})$ and $\bar{b}' = p(\bar{b})$. Then the stable categories $\underline{\mathrm{mod}}\,\Lambda$ and $\underline{\mathrm{mod}}\,\Lambda'$ are equivalent.*

In the case of Theorem 7, we will show that this problem is true if K is an algebraically closed field.

Theorem 8. *Let Λ and Λ' be self-injective algebras over algebraically closed field, and $\Phi : \underline{\mathrm{mod}}\,\Lambda \xrightarrow{\sim} \underline{\mathrm{mod}}\,\Lambda'$ be a stable equivalence defined in Theorem 7. If Λ is symmetric, then so is Λ'.*

References

[1] M. Auslander, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Adv. Math. **36**, Cambridge 1995.

[2] D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Math. Soc. Lecture Notes **119**, University Press, Cambridge (1988).

[3] R. Hartshorne, *Residues and Duality*, Lecture Notes in Mathematics **20**, Springer, Berlin (1966).
[4] B. Keller and D. Vossieck, *Sous les catégories dérivées*, C. R. Acad. Soc. Paris, **305** (1987), 225–228.
[5] Y. Ohnuki, *Stable equivalence induced by a socle equivalence*, Osaka J. Math. **39** (2002), 259–266.
[6] ———, K. Takeda and K. Yamagata, *Symmetric Hochschild extension algebras*, Colloquium Math. **80** (1999), 155–174.
[7] Z. Pogorzały, *Invariance of Hochschild cohomology algebras under stable equivalences of Morita type*, J. Math. Soc. Japan **53** (2001), 913–918.
[8] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Alg. **61** (1989), 303–317.
[9] C. C. Xi, *On the representation dimension of finite dimensional algebras*, J. Algebra **226** (2000), 332–346.
[10] K. Yamagata, *Frobenius algebras*, Handbook of Algebra, Elsevier Science B. V. Vol 1 (1996), 841–887.

Tokyo University of Agriculture and Technology
2-24-16 Nakacho, Koganei
Tokyo 184-8588, JAPAN
E-mail address: ohnuki@cc.tuat.ac.jp