CASTELNUOVO-MUMFORD REGULARITY FOR COMPLEXES AND RELATED TOPICS

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Abstract. Let A be a noetherian AS regular Koszul quiver algebra (if A is commutative, it is essentially a polynomial ring), and $\operatorname{gr} A$ the category of finitely generated graded left A-modules. Following Jørgensen, we define the Castelnuovo-Mumford regularity $\operatorname{reg}(M^\bullet)$ of a complex $M^\bullet \in D^b(\operatorname{gr} A)$ in terms of the local cohomologies or the minimal projective resolution of $M^\bullet$. Let $A^!$ be the quadratic dual ring of A. Then $A^!$ is selfinjective and Koszul (e.g., if A is the polynomial ring $k[x_1, \dots, x_d]$, then $A^!$ is the exterior algebra $E = \bigwedge \langle y_1, \dots, y_d \rangle$). For the Koszul duality functor $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$, we have $\operatorname{reg}(M^\bullet) = \max\{\, i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \,\}$. As an application, we refine results of Martinez-Villa and Zacharia on *weakly Koszul modules* over $A^!$ (especially, over E).

1. Introduction

Let $A := \bigoplus_{i \geq 0} A_i$ be a noetherian AS regular Koszul quiver algebra over a field k. Such a quiver algebra (with relation) has been studied by Martinez-Villa and coworkers (c.f. [6, 9, 10, 11]). And a *connected* (i.e., $A_0 = k$) AS regular algebra is very important in non-commutative algebraic geometry (c.f. [18]). If A is commutative and connected, it is a polynomial ring $k[x_1, \dots, x_d]$ with $\deg x_i = 1$ for each i.

Let $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) be the category of graded left (resp. right) A-modules, and $\operatorname{gr} A$ (resp. $\operatorname{gr} A^{\mathsf{op}}$) its full subcategory consisting of finitely generated modules. Set $\mathfrak{r} := \bigoplus_{i \geq 1} A_i$ to be the graded Jacobson radical. We have the left exact functor $\Gamma_{\mathfrak{r}} : \operatorname{Gr} A \to \operatorname{Gr} A$ defined by $\Gamma_{\mathfrak{r}}(M) = \{\, x \in M \mid \mathfrak{r}^n x = 0 \text{ for } n \gg 0 \,\}$, and its right derived functor $R\Gamma_{\mathfrak{r}} : D^b(\operatorname{Gr} A) \to D^b(\operatorname{Gr} A)$. For $M^\bullet \in D^b(\operatorname{Gr} A)$, the i^{th} cohomology of $R\Gamma_{\mathfrak{r}}(M^\bullet)$ is denoted by $H^i_{\mathfrak{r}}(M^\bullet)$. Similarly, we have the corresponding functors $\Gamma_{\mathfrak{r}^{\mathsf{op}}}$, $R\Gamma_{\mathfrak{r}^{\mathsf{op}}}$, and $H^i_{\mathfrak{r}^{\mathsf{op}}}$ for graded *right* A-modules. When A is a polynomial ring $k[x_1, \dots, x_d]$, $H^i_{\mathfrak{r}}(-)$ is known as the *local cohomology module* with support in the graded maximal ideal $\mathfrak{r}$.

We have a bounded cochain complex $\mathcal{D}^\bullet$ of graded A-A bimodules which gives duality functors $R\operatorname{Hom}_A(-, \mathcal{D}^\bullet) : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^{\mathsf{op}})$ and $R\operatorname{Hom}_{A^{\mathsf{op}}}(-, \mathcal{D}^\bullet) : D^b(\operatorname{gr} A^{\mathsf{op}}) \to D^b(\operatorname{gr} A)$. These functors are quasi-inverse of each other. Moreover, we have "local duality theorem"

$$R\underline{\operatorname{Hom}}_A(-, \mathcal{D}^\bullet) \cong R\Gamma_{\mathfrak{r}}(-)^\vee \quad \text{and} \quad R\underline{\operatorname{Hom}}_{A^{\mathsf{op}}}(-, \mathcal{D}^\bullet) \cong R\Gamma_{\mathfrak{r}^{\mathsf{op}}}(-)^\vee,$$

where $(-)^\vee$ stands for the graded k-dual. This is a quiver algebra version of [18].

For $M^\bullet \in D^b(\operatorname{gr} A)$ and $i, j \in \mathbb{Z}$, set $\beta^i_j(M^\bullet) := \dim_k \underline{\operatorname{Ext}}_A^{-i}(M^\bullet, A/\mathfrak{r})_{-j}$. Of course, $\beta^i_j(-)$ measures the "size" of a minimal projective resolution. Using the above duality, we can generalize a well-know result of Eisenbud-Goto [5] concerning graded modules over a polynomial ring.

This note is basically a summary of [17] which has been accepted for publication in *J. Pure Appl. Algebra*.

Definition-Theorem. (c.f. Jørgensen, [8]) *For $M^\bullet \in D^b(\operatorname{gr} A)$, we have*

$$\operatorname{reg}(M^\bullet) := \sup\{\, i+j \mid H^i_{\mathfrak{r}}(M^\bullet)_j \neq 0 \,\} = \sup\{\, i+j \mid \beta^i_j(M^\bullet) \neq 0 \,\} < \infty.$$

We call this value the "Castelnuovo-Mumford regularity" of $M^\bullet$.

For $M^\bullet \in D^b(\operatorname{gr} A)$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and the differential maps are zero. Then $\operatorname{reg}(\mathcal{H}(M^\bullet)) \geq \operatorname{reg}(M^\bullet)$. The difference $\operatorname{reg}(\mathcal{H}(M^\bullet)) - \operatorname{reg}(M^\bullet)$ is a theme of the latter half of this note.

Let $A^!$ be the quadratic dual ring of A. Then $A^!$ is finite dimensional, Koszul and self-injective by [10]. (e.g., If A is the polynomial ring $k[x_1, \dots, x_d]$, then $A^!$ is the exterior algebra $\bigwedge \langle y_1, \dots, y_d \rangle$). The Koszul duality functors $\mathcal{F} : D^b(\operatorname{gr} A^!) \to D^b(\operatorname{gr} A)$ and $\mathcal{G} : D^b(\operatorname{gr} A) \to D^b(\operatorname{gr} A^!)$ give an equivalence $D^b(\operatorname{gr} A) \cong D^b(\operatorname{gr} A^!)$ (c.f. [2]). We have

$$\operatorname{reg}(M^\bullet) = \max\{\, i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0 \,\}.$$

For $N \in \operatorname{gr} A^!$ and $n \in \mathbb{Z}$, $N_{\langle n \rangle}$ denotes the submodule of N generated by the degree n component N_n. We say N is *weakly Koszul*, if $N_{\langle n \rangle}$ has an *n-linear projective resolution* (i.e., $\beta^i_j(N_{\langle n \rangle}) \neq 0 \Rightarrow i+j = n$) for all n. Martinez-Villa and Zacharia [11] proved that the i^{th} syzygy $\Omega_i(N)$ of $N \in \operatorname{gr} A^!$ is weakly Koszul for $i \gg 0$. Of course, the same is true for $N \in \operatorname{gr} (A^!)^{\mathsf{op}}$. Set $\operatorname{lpd}(N) := \min\{\, i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul} \,\}$.

Theorem. *Let $N \in \operatorname{gr} A^!$, and $N' := \underline{\operatorname{Hom}}_{A^!}(N, A^!) \in \operatorname{gr} (A^!)^{\mathsf{op}}$ its dual. Then*

$$\operatorname{lpd}(N') = \operatorname{reg}(\, \mathcal{H} \circ \mathcal{F}(N)\,).$$

If $A^!$ is the exterior algebra $E = \bigwedge \langle y_1, \dots, y_d \rangle$, we have an upper bound of $\operatorname{lpd}(N)$ depending only on $\max\{\, \dim_k N_i \mid i \in \mathbb{Z} \,\}$ and d. This bound gives huge numbers, and must be very far from optimal. On the other hand, we have $\operatorname{lpd}(E/J) \leq \min\{1, d-2\}$ for a monomial ideal J of E. This slightly improves a result of Herzog and Römer.

2. Preliminaries

First, we sketch basic properties of an algebra of a quiver with relations.

Let Q be a finite quiver. That is, $Q = (Q_0, Q_1)$ is a finite oriented graph, where Q_0 is the set of vertices and Q_1 is the set of arrows. The path algebra kQ is a positively graded algebra with grading given by the lengths of paths. Let J be the graded Jacobson radical of kQ (i.e., the ideal generated by all arrows). If $I \subset J^2$ is a graded ideal, we say $A = kQ/I$ is a *graded quiver algebra*. Of course, $A = \bigoplus_{i \geq 0} A_i$ is a graded ring. The subalgebra A_0 is a product of copies of the field k, one copy for each element of Q_0. If $A_0 = k$ (i.e., Q has only one vertex), we say A is *connected*. If a graded algebra $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = k$ is generated by R_1 as a k-algebra and $\dim_k R_1 < \infty$, then it can be regarded as a graded quiver algebra. Set $\mathfrak{r} := \bigoplus_{i \geq 1} A_i$. Unless otherwise specified, we assume that A is left and right noetherian throughout this note.

Let $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) be the category of graded left (resp. right) A-modules, and $\operatorname{gr} A$ (resp. $\operatorname{gr} A^{\mathsf{op}}$) its full subcategory consisting of finitely generated modules. Since A is noetherian, $\operatorname{gr} A$ and $\operatorname{gr} A^{\mathsf{op}}$ are abelian categories. In the sequel, we will define several concepts for $\operatorname{Gr} A$, but the corresponding concepts for $\operatorname{Gr} A^{\mathsf{op}}$ can be defined in the similar way.

The n^{th} shift $M(n)$ of $M = \bigoplus_{i \in \mathbb{Z}} M_i \in \operatorname{Gr} A$ is defined by $M(n)_i = M_{n+i}$. Set $\iota(M) := \inf\{ i \mid M_i \neq 0 \}$.

For $v \in Q_0$, we have the idempotent e_v of A associated with v. Note that $1 = \sum_{v \in Q_0} e_v$. Set $P_v := Ae_v$ and ${}_vP := e_vA$. Then we have ${}_AA = \bigoplus_{v \in Q_0} P_v$ and $A_A = \bigoplus_{v \in Q_0} ({}_vP)$. Each P_v and ${}_vP$ are indecomposable projectives. Conversely, any indecomposable projective in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$) is isomorphic to P_v (resp. ${}_vP$) for some $v \in Q_0$ up to degree shifting. Set $k_v := P_v/(\mathfrak{r}P_v)$ and ${}_vk := {}_vP/({}_vP\,\mathfrak{r})$. Each k_v and ${}_vk$ are simple.

Let $C^b(\operatorname{Gr} A)$ be the category of bounded cochain complexes in $\operatorname{Gr} A$, and $D^b(\operatorname{Gr} A)$ its derived category. For a complex $M^\bullet$ and an integer p, let $M^\bullet[p]$ be the p^{th} translation of $M^\bullet$. That is, $M^\bullet[p]$ is a complex with $M^i[p] = M^{i+p}$. A module M can be regarded as a complex $\cdots \to 0 \to M \to 0 \to \cdots$ with M at the 0^{th} term.

For $M, N \in \operatorname{Gr} A$, set $\underline{\operatorname{Hom}}_A(M,N) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$ to be a graded k-vector space with $\underline{\operatorname{Hom}}_A(M,N)_i = \operatorname{Hom}_{\operatorname{Gr} A}(M, N(i))$. Similarly, we can also define $\underline{\operatorname{Hom}}^\bullet_A(M^\bullet, N^\bullet)$, $\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, N^\bullet)$, and $\underline{\operatorname{Ext}}^i_A(M^\bullet, N^\bullet)$ for $M^\bullet, N^\bullet \in D^b(\operatorname{Gr} A)$.

If V is a k-vector space, V^* denotes the dual space $\operatorname{Hom}_k(V,k)$. For $M \in \operatorname{Gr} A$ (resp. $M \in \operatorname{Gr} A^{\mathsf{op}}$), $M^\vee := \bigoplus_{i \in \mathbb{Z}} (M_i)^*$ has a graded *right* (resp. *left*) A-module structure given by $(fa)(x) = f(ax)$ (resp. $(af)(x) = f(xa)$) and $(M^\vee)_i = (M_{-i})^*$. If W is a graded A-A bimodule, then so is $W^\vee$. Note that $I_v := ({}_vP)^\vee$ (resp. ${}_vI := (P_v)^\vee$) is injective in $\operatorname{Gr} A$ (resp. $\operatorname{Gr} A^{\mathsf{op}}$). Moreover, I_v and ${}_vI$ are graded injective hulls of k_v and ${}_vk$ respectively. In particular, the A-A bimodule $A^\vee$ is injective both in $\operatorname{Gr} A$ and in $\operatorname{Gr} A^{\mathsf{op}}$.

Let W be a graded A-A-bimodule (we mainly concern the cases $W = A$ or $W = A^\vee$). If $M \in \operatorname{Gr} A$, we can regard $\underline{\operatorname{Hom}}_A(M, W)$ as a graded *right* A-module by $(fa)(x) = f(x)a$. We have a natural isomorphism $\underline{\operatorname{Hom}}_A(M, A^\vee) \cong M^\vee$. We can also define $\mathbf{R}\underline{\operatorname{Hom}}_A(M^\bullet, W) \in D^b(\operatorname{Gr} A^{\mathsf{op}})$ and $\underline{\operatorname{Ext}}^i_A(M^\bullet, W) \in \operatorname{Gr} A^{\mathsf{op}}$ for $M^\bullet \in D^b(\operatorname{Gr} A)$.

Let $P^\bullet$ be a bounded complex in $\operatorname{gr} A$ such that each P^i is projective. We say $P^\bullet$ is *minimal* if $\partial(P^i) \subset \mathfrak{r}P^{i+1}$ for all i. Any complex $M^\bullet \in C^b(\operatorname{gr} A)$ has a minimal projective resolution, which is unique up to isomorphism. We denote a graded module $A/\mathfrak{r}$ by A_0. Set $\beta^i_j(M^\bullet) := \dim_k \underline{\operatorname{Ext}}^{-i}_A(M^\bullet, A_0)_{-j}$. Let $P^\bullet$ be a minimal projective resolution of $M^\bullet$, and $P^i := \bigoplus_{l=1}^m T^{i,l}$ an indecomposable decomposition. Then we have

$$\beta^i_j(M^\bullet) = \#\{ l \mid T^{i,l}(j) \cong P_v \text{ for some } v \}.$$

Definition 1. Let A be a graded quiver algebra. We say A is *Artin-Schelter regular* (AS-regular, for short), if

- A has finite global dimension d.
- $\underline{\operatorname{Ext}}^i_A(k_v, A) = \underline{\operatorname{Ext}}^i_{A^{\mathsf{op}}}({}_vk, A) = 0$ for all $i \neq d$ and all $v \in Q_0$.
- There are a permutation δ on Q_0 and an integer n_v for each $v \in Q_0$ such that $\underline{\operatorname{Ext}}^d_A(k_v, A) \cong {}_{\delta(v)}k(n_v)$ (equivalently, $\underline{\operatorname{Ext}}^d_{A^{\mathsf{op}}}({}_vk, A) \cong k_{\delta^{-1}(v)}(n_v)$) for all v.

Remark 2. The AS regularity is a very important concept in non-commutative algebraic geometry (see for example [18]). But there many authors assume the connectedness of A. We also remark that Martinez-Villa and coworkers called the rings in Definition 1 *generalized Auslander regular algebras*.

Definition 3. For an integer $l \in \mathbb{Z}$, we say $M^\bullet \in \operatorname{gr} A$ has an *l-linear (projective) resolution*, if $\beta^i_j(M^\bullet) = 0$ for all i, j with $i + j \neq l$. If $M^\bullet$ has an l-linear resolution for some l, we say $M^\bullet$ has a *linear resolution.*

Definition 4. We say A is *Koszul*, if the graded left A-module A_0 has a linear resolution. (Note that $A_0 \cong \bigoplus_{v \in Q_0} k_v$.)

In the above definition, we can regard A_0 as a right A-module (we get the equivalent definition). The next fact is easy to prove.

Lemma 5. *If A is AS-regular, Koszul, and has global dimension d, then $\underline{\operatorname{Ext}}^d_A(k_v, A) \cong {}_{\delta(v)}k(d)$ and $\underline{\operatorname{Ext}}^d_{A^{\mathsf{op}}}({}_vk, A) \cong k_{\delta^{-1}(v)}(d)$ for all $v \in Q_0$. Here δ is the permutation of Q_0 given in Definition 1.*

In the rest of this paper, A is always a noetherian AS-regular Koszul quiver algebra of global dimension d.

Example 6. (1) A polynomial ring $k[x_1, \dots , x_d]$ is clearly a noetherian AS-regular Koszul (quiver) algebra of global dimension d.

(2) Let $k\langle x_1, \dots , x_d\rangle$ be the free associative algebra, and $(q_{i,j})$ a $d\times d$ matrix with entries in k satisfying $q_{i,j}q_{j,i} = q_{i,i} = 1$ for all i, j. Then $A = k\langle x_1, \dots , x_n\rangle/(\, x_jx_i - q_{i,j}x_ix_j \mid 1 \leq i, j \leq d\,)$ is a noetherian AS-regular Koszul algebra with global dimension d. This fact must be well-known to the specialist, but we will sketch a proof here. Since $x_1, ..., x_d \in A_1$ form a regular normalizing sequence with $k = A/(x_1, \dots , x_d)$, A is a noetherian ring with a balanced dualizing complex by [12, Lemma 7.3]. We can construct a free resolution of $k = A/\mathfrak{r}$, which is a "q-analog" of the Koszul complex of a polynomial ring $k[x_1, \dots , x_d]$. So A is Koszul and has global dimension d. Since A has finite global dimension and admits a balanced dualizing complex, it is AS-regular (c.f. [12, Remark 3.6 (3)]).

(3) For examples of non-connected AS regular algebras, see [6].

For $M \in \operatorname{Gr} A$, set

$$\Gamma_{\mathfrak{r}}(M) = \varinjlim \underline{\operatorname{Hom}}_A(A/\mathfrak{r}^n, M) = \{\, x \in M \mid A_n\, x = 0 \text{ for } n \gg 0 \,\} \in \operatorname{Gr} A.$$

Then $\Gamma_{\mathfrak{r}}(-)$ gives a left exact functor from $\operatorname{Gr} A$ to itself. So we have a right derived functor $\mathbf{R}\Gamma_{\mathfrak{r}} : D^b(\operatorname{Gr} A) \to D^b(\operatorname{Gr} A)$. For $M^\bullet \in D^b(\operatorname{Gr} A)$, $H^i_{\mathfrak{r}}(M^\bullet)$ denotes the i^{th} cohomology of $\mathbf{R}\Gamma_{\mathfrak{r}}(M^\bullet)$. Similarly, we can define $\mathbf{R}\Gamma_{\mathfrak{r}^{\mathsf{op}}}$ and $H^i_{\mathfrak{r}^{\mathsf{op}}}$ for $D^b(\operatorname{Gr} A^{\mathsf{op}})$ in the same way. If M is an A-A bimodule, $H^i_{\mathfrak{r}}(M)$ and $H^i_{\mathfrak{r}^{\mathsf{op}}}(M)$ are also.

Since A is AS regular, we have $\mathbf{R}\Gamma_{\mathfrak{r}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A)$. By the same argument as [18, Proposition 4.4], we also have $\mathbf{R}\Gamma_{\mathfrak{r}}(A) \cong A^\vee(d)[-d]$ in $D^b(\operatorname{gr} A^{\mathsf{op}})$. It does not mean that $H^d_{\mathfrak{r}}(A) \cong A^\vee(d)$ as A-A bimodules. But there is an A-A bimodule L such that $L \otimes_A H^d_{\mathfrak{r}}(A) \cong A^\vee(d)$ as A-A bimodules. Here the underlying graded additive group of L is A, but the bimodule structure is give by $A \times L \times A \ni (a, l, b) \mapsto \phi(a)lb \in A = L$ for a (fixed) graded k-algebra automorphism ϕ of A. In particular, $L \cong A$ as left A-modules and as right A-modules (separately). If A is commutative, then ϕ is the identity map.

Set $L' \cong \underline{\operatorname{Hom}}_A(L, A)$ and $\mathcal{D}^\bullet := L'(-d)[d]$. Note that $\mathcal{D}^\bullet$ belongs both $D^b(\operatorname{gr} A)$ and $D^b(\operatorname{gr} A^{\mathsf{op}})$. We have $H^i_{\mathfrak{r}}(\mathcal{D}^\bullet) = H^i_{\mathfrak{r}^{\mathsf{op}}}(\mathcal{D}^\bullet) = 0$ for all $i \neq 0$ and $H^0_{\mathfrak{r}}(\mathcal{D}^\bullet) \cong H^0_{\mathfrak{r}^{\mathsf{op}}}(\mathcal{D}^\bullet) \cong A^\vee$ as A-A bimodules by the same argument as [18, §4]. Thus (an injective resolution of) $\mathcal{D}^\bullet$ is a *balanced dualizing complex* of A in the sense of [18]. It is easy to check that $\mathbf{R}\underline{\operatorname{Hom}}_A(-, \mathcal{D}^\bullet)$

and $\mathbf{R}\underline{\mathrm{Hom}}_{A^{\mathsf{op}}}(-,\mathcal{D}^\bullet)$ give duality functors between $D^b(\mathrm{gr}\,A)$ and $D^b(\mathrm{gr}\,A^{\mathsf{op}})$, which are quasi-inverse of each other.

Theorem 7 (c.f. Yekutieli [18] and Martinez-Villa [9]). *For $M^\bullet \in D^b(\mathrm{gr}\,A)$, we have*

$$\mathbf{R}\Gamma_{\mathfrak{r}}(M^\bullet)^\vee \cong \mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet,\mathcal{D}^\bullet). \quad \textit{In particular,} \quad (H^i_{\mathfrak{r}}(M^\bullet)_j)^* \cong \underline{\mathrm{Ext}}_A^{-i}(M^\bullet,\mathcal{D}^\bullet)_{-j}.$$

The above result was proved by Yekutieli in the connected case. (In some sense, Martinez-Villa proved a more general result than ours, but he did not concern complexes.) The proof of [18, Theorem 4.18] also works in our case.

Definition 8 (Jørgensen, [8]). For $M^\bullet \in D^b(\mathrm{gr}\,A)$, we say

$$\mathrm{reg}(M^\bullet) := \sup\{\, i+j \mid H^i_{\mathfrak{r}}(M^\bullet)_j \neq 0 \,\}$$

is the *Castelnuovo-Mumford regularity* of $M^\bullet$.

By Theorem 7 and the fact that $\mathbf{R}\underline{\mathrm{Hom}}_A(M^\bullet,\mathcal{D}^\bullet) \in D^b(\mathrm{gr}\,A^{\mathsf{op}})$, we have $\mathrm{reg}(M^\bullet) < \infty$ for all $M^\bullet \in D^b(\mathrm{gr}\,A)$.

Theorem 9 (Jørgensen, [8]). *If $M^\bullet \in C^b(\mathrm{gr}\,A)$, then*

$$\mathrm{reg}(M^\bullet) = \max\{\, i+j \mid \beta^i_j(M^\bullet) \neq 0 \,\}. \tag{2.1}$$

When A is a polynomial ring and $M^\bullet$ is a module, the above theorem is a fundamental result obtained by Eisenbud and Goto [5]. In the non-commutative case, under the assumption that A is connected but not necessarily regular, this has been proved by Jørgensen [8]. (If A is not regular, we have $\mathrm{reg}(A) > 0$ in many cases. So one has to assume that $\mathrm{reg}\,A = 0$ there.) In our case (i.e., A is AS-regular), we have a much simpler proof. So we will give it here. This proof is also different from one given in [5].

Proof. Set $Q^\bullet := \underline{\mathrm{Hom}}^\bullet_A(P^\bullet, L'(-d)[d]\,)$. Here $P^\bullet$ is a minimal projective resolution of $M^\bullet$, and L' is the A-A bimodule given in the construction of the dualizing complex $\mathcal{D}^\bullet$. Note that $\underline{\mathrm{Hom}}_A(P_v, L') \cong {}_{\delta^{-1}(v)}P$ for all $v \in Q_0$. Let s be the right hand side of (2.1), and l the minimal integer with the property that $\beta^l_{s-l}(M^\bullet) \neq 0$. Then $\iota(Q^{-d-l}) = l-s+d$, and $\iota(Q^{-d-l+1}) \geq l-s+d$. Since $Q^\bullet$ is a minimal complex, we have

$$0 \neq H^{-d-l}(Q^\bullet)_{l-s+d} = \underline{\mathrm{Ext}}_A^{-d-l}(M^\bullet,\mathcal{D}^\bullet)_{l-s+d} = (H^{d+l}_{\mathfrak{r}}(M^\bullet)_{-l+s-d})^*.$$

Thus $\mathrm{reg}(M^\bullet) \geq s$. The opposite inequality can be proved similarly and more easily. □

3. Koszul duality

In this section, we study the relation between the Castelnuovo-Mumford regularity of complexes and the Koszul duality. For precise information of this duality, see [2, §2].

Recall that $A = kQ/I$ is a graded quiver algebra over a finite quiver Q. Let Q^{op} be the *opposite quiver* of Q. That is, $Q_0^{\mathsf{op}} = Q_0$ and there is a bijection from Q_1 to Q_1^{op} which sends an arrow $\alpha : v \to u$ in Q_1 to the arrow $\alpha^{\mathsf{op}} : u \to v$ in Q_1^{op}. Consider the bilinear form $\langle -,- \rangle : (kQ)_2 \times (kQ^{\mathsf{op}})_2 \to A_0$ defined by

$$\langle \alpha\beta, \gamma^{\mathsf{op}}\delta^{\mathsf{op}} \rangle = \begin{cases} e_v & \text{if } \alpha=\delta \text{ and } \beta=\gamma, \\ 0 & \text{otherwise} \end{cases}$$

for all $\alpha, \beta, \gamma, \delta \in Q_1$. Here $v \in Q_0$ is the vertex with $\beta \in Ae_v$. Let $I^\perp \subset kQ^{\mathsf{op}}$ be the ideal generated by $\{\, y \in (kQ^{\mathsf{op}})_2 \mid \langle x, y\rangle = 0 \text{ for all } x \in I_2 \,\}$. We say $kQ^{\mathsf{op}}/I^\perp$ is the *quadratic dual ring* of A, and denote it by $A^!$. Clearly, $(A^!)_0 = A_0$. Since A is Koszul, so is $A^!$. Since A is AS regular, $A^!$ is a finite dimensional selfinjective algebra with $A = \bigoplus_{i=0}^d A_i$ by [10, Theorem 5.1]. If A is a polynomial ring, then $A^!$ is the exterior algebra $\bigwedge(A_1)^*$.

Let V be a finitely generated left A_0-module. Then $\mathrm{Hom}_{A_0}(A^!, V)$ is a graded *left* $A^!$-module with $(af)(a') = f(a'a)$ and $\mathrm{Hom}_{A_0}(A^!, V)_i = \mathrm{Hom}_{A_0}((A^!)_{-i}, V)$. Since $A^!$ is selfinjective, we have $\mathrm{Hom}_{A_0}(A^!, A_0) \cong A^!(d)$. Hence $\mathrm{Hom}_{A_0}(A^!, V)$ is a projective (and injective) left $A^!$-module for all V. If V has degree i (e.g., $V = M_i$ for some $M \in \mathrm{gr}\, A$), then we set $\mathrm{Hom}_{A_0}(A^!, V)_j = \mathrm{Hom}_{A_0}(A^!_{-j-i}, V)$.

For $M^\bullet \in C^b(\mathrm{gr}\, A)$, let $\mathcal{G}(M^\bullet) := \mathrm{Hom}_{A_0}(A^!, M^\bullet) \in C^b(\mathrm{gr}\, A^!)$ be the total complex of the double complex with $\mathcal{G}(M^\bullet)^{i,j} = \mathrm{Hom}_{A_0}(A^!, M^i_j)$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(f)(x) = \sum_{\alpha \in Q_1} \alpha f(\alpha^{\mathsf{op}} x), \qquad d''(f)(x) = \partial_{M^\bullet}(f(x))$$

for $f \in \mathrm{Hom}_{A_0}(A^!, M^i_j)$ and $x \in A^!$. The grading of $\mathcal{G}(M^\bullet)$ is given by

$$\mathcal{G}(M^\bullet)^p_q := \bigoplus_{p=i+j,\, q=-l-j} \mathrm{Hom}_{A_0}((A^!)_l, M^i_j).$$

Similarly, for a complex $N^\bullet \in C^b(\mathrm{gr}\, A^!)$, we can define a new complex $\mathcal{F}(N^\bullet) := A \otimes_{A_0} N^\bullet \in C^b(\mathrm{gr}\, A)$ as the total complex of the double complex with $\mathcal{F}(N^\bullet)^{i,j} = A \otimes_{A_0} N^i_j$ whose vertical and horizontal differentials d' and d'' are defined by

$$d'(a \otimes x) = \sum_{\alpha \in Q_1} a\alpha \otimes \alpha^{\mathsf{op}} x, \qquad d''(a \otimes x) = a \otimes \partial_{N^\bullet}(x)$$

for $a \otimes x \in A \otimes_{A_0} N^i$. The gradings of $\mathcal{F}(N^\bullet)$ is given by

$$\mathcal{F}(N^\bullet)^p_q := \bigoplus_{p=i+j,\, q=l-j} A_l \otimes_{A_0} N^i_j.$$

Clearly, each term of $\mathcal{F}(N^\bullet)$ is a projective A-module. For a module $N \in \mathrm{gr}\, A^!$, $\mathcal{F}(N)$ is a minimal complex. Hence we have

$$\beta^i_j(\mathcal{F}(N)) = \begin{cases} \dim_k N_i & \text{if } i + j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The operations $\mathcal{F}$ and $\mathcal{G}$ define functors $\mathcal{F} : D^b(\mathrm{gr}\, A^!) \to D^b(\mathrm{gr}\, A)$ and $\mathcal{G} : D^b(\mathrm{gr}\, A) \to D^b(\mathrm{gr}\, A^!)$, and they give an equivalence $D^b(\mathrm{gr}\, A) \cong D^b(\mathrm{gr}\, A^!)$ of triangulated categories. This equivalence is called the *Koszul duality.* When A is a polynomial ring, this equivalence is called *Bernstein-Gel'fand-Gel'fand correspondence.* See, for example, [4].

Proposition 10 (c.f. [4, Proposition 2.3])**.** *In the above situation, we have*

$$\beta^i_j(M^\bullet) = \dim_k H^{i+j}(\mathcal{G}(M^\bullet))_{-j}.$$

Proof.

$$\begin{aligned}
\underline{\mathrm{Ext}}^i_{A^!}(A_0, N^\bullet)_j &\cong \mathrm{Hom}_{D^b(\mathrm{gr}\,A^!)}(A_0, N^\bullet[i](j)) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{gr}\,A)}(\,\mathcal{F}(A_0),\, \mathcal{F}(N^\bullet[i](j))\,) \\
&\cong \mathrm{Hom}_{D^b(\mathrm{gr}\,A)}(\,A,\, \mathcal{F}(N^\bullet)[i+j](-j)\,) \\
&\cong H^{i+j}(\mathcal{F}(N^\bullet))_{-j}.
\end{aligned}$$

□

The next result immediately follows from Theorem 9 and Proposition 10.

Corollary 11. $\mathrm{reg}(M^\bullet) = \max\{\, i \mid H^i(\mathcal{G}(M^\bullet)) \neq 0\,\}$.

For $M^\bullet \in D^b(\mathrm{gr}\,A)$, set $\mathcal{H}(M^\bullet)$ to be the complex such that $\mathcal{H}(M^\bullet)^i = H^i(M)$ for all i and all differential maps are zero. By a spectral sequence argument, we see that

$$\mathrm{reg}(\mathcal{H}(M^\bullet)) \geq \mathrm{reg}(M^\bullet). \tag{3.1}$$

In the next section, we will see that the difference $\mathrm{reg}(M^\bullet) - \mathrm{reg}(\mathcal{H}(M^\bullet))$ can be arbitrary large. For $N^\bullet \in D^b(\mathrm{gr}\,A^!)$, we can define $\mathcal{H}(N^\bullet)$ is the same way.

We can refine Proposition 10 using the notion of *linear strands* of projective resolutions, which was introduced by Eisenbud et. al. ([4, §3]). Let $P^\bullet$ be a *minimal* projective resolution of $M^\bullet \in D^b(\mathrm{gr}\,A)$. Consider the decomposition $P^i := \bigoplus_{j\in\mathbb{Z}} P^{i,j}$ such that any indecomposable summand of $P^{i,j}$ is isomorphic to a summand of $A(-j)$. For an integer l, we define the *l-linear strand* $\mathrm{proj.\,lin}_l(M^\bullet)$ of a projective resolution of $M^\bullet$ as follows: The term $\mathrm{proj.\,lin}_l(M^\bullet)^i$ of cohomological degree i is $P^{i,l-i}$ and the differential $P^{i,l-i} \to P^{i+1,l-i-1}$ is the corresponding component of the differential $P^i \to P^{i+1}$ of $P^\bullet$. So the differential of $\mathrm{proj.\,lin}_l(M^\bullet)$ is represented by a matrix whose entries are elements in A_1. Set $\mathrm{proj.\,lin}(M^\bullet) := \bigoplus_{l\in\mathbb{Z}} \mathrm{proj.\,lin}_l(M^\bullet)$. Clearly, $\beta^i_j(M^\bullet) = \beta^i_j(\mathrm{proj.\,lin}(M^\bullet))$ for all i, j.

Proposition 12 (c.f. [4, Corollary 3.6])**.** *For $N^\bullet \in D^b(\mathrm{gr}\,A^!)$, we have*

$$\mathrm{proj.\,lin}_l(\mathcal{F}(N^\bullet)) = \mathcal{F}(H^l(N^\bullet))[-l], \quad \textit{in particular,} \quad \mathrm{proj.\,lin}(\mathcal{F}(N^\bullet)) = \mathcal{F}(\mathcal{H}(N^\bullet)).$$

4. Weakly Koszul Modules

Let B be a noetherian Koszul algebra with the graded Jacobson radical $\mathfrak{r}$. For $M \in \mathrm{gr}\,B$ and $i \in \mathbb{Z}$, $M_{\langle i\rangle}$ denotes the submodule of M generated by its degree i component M_i. The next result naturally appears in the study of Koszul algebras, and might be a folk-theorem (see [17] for further information).

Proposition 13. *In the above situation, the following are equivalent.*

1. $M_{\langle i\rangle}$ *has a linear projective resolution for all i.*
2. $H^i(\mathrm{proj.\,lin}(M)) = 0$ *for all $i \neq 0$.*
3. $\mathrm{gr}_{\mathfrak{r}}\, M := \bigoplus_{i=0}^{\infty} \mathfrak{r}^{i-1}M/\mathfrak{r}^i M$ *has a linear resolution as a $B\,(\cong \mathrm{gr}_{\mathfrak{r}}\, B)$-modules.*

Definition 14 (Martinez-Villa et.al., c.f. [11])**.** We say $M \in \mathrm{gr}\,B$ is *weakly Koszul*, if it satisfies the equivalent conditions of Proposition 13.

If $M \in \operatorname{gr} B$ has a linear resolution, then it is weakly Koszul. Moreover, if M is weakly Koszul, then the i^{th} syzygy $\Omega_i(M)$ is also for all $i \geq 1$.

Let A be a noetherian AS-regular Koszul quiver algebra of global dimension d, and $A^!$ its quadratic dual, as in the previous sections.

Theorem 15 (Martinez-Villa and Zacharia, [11])**.** *If $N \in \operatorname{gr} A^!$ (or $N \in \operatorname{gr}(A^!)^{\mathsf{op}}$), then the i^{th} syzygy $\Omega_i(N)$ is weakly Koszul for $i \gg 0$.*

Definition 16 (Herzog et. al., [7, 14])**.** For $0 \neq N \in \operatorname{gr} A^!$ (or $N \in \operatorname{gr}(A^!)^{\mathsf{op}}$), set

$$\operatorname{lpd}(N) := \inf\{\, i \in \mathbb{N} \mid \Omega_i(N) \text{ is weakly Koszul} \,\}.$$

Remark 17. Herzog and Iyengar ([7]) studied the invariant lpd over noetherian commutative Koszul algebras. Among other things, they proved that $\operatorname{lpd}(M)$ is always finite over some "nice" rings (e.g., graded complete intersections which are Koszul).

The next result follows from Corollary 11 and Proposition 12.

Theorem 18. *Let $N \in \operatorname{gr} A^!$, and $N' := \operatorname{Hom}_{A^!}(N, A^!) \in \operatorname{gr}(A^!)^{\mathsf{op}}$ its dual. Then we have*

$$\begin{aligned} \operatorname{lpd}(N') &= \operatorname{reg}(\,\mathcal{H} \circ \mathcal{F}(N)\,) \\ &= \max\{\, \operatorname{reg}(H^i(\mathcal{F}(N))) + i \mid i \in \mathbb{Z} \,\}. \end{aligned}$$

Note that $\operatorname{reg}(\mathcal{H} \circ \mathcal{F}(N)) \geq \operatorname{reg}(\mathcal{F}(N)) = \max\{\, i \mid H^i(\mathcal{G} \circ \mathcal{F}(N)) \neq 0 \,\} = 0$ by the inequality (3.1) and Corollary 11.

If $\operatorname{lpd}(N) \geq 1$ for some $N \in \operatorname{gr} A^!$, then $\sup\{\, \operatorname{lpd}(L) \mid L \in \operatorname{gr} A^! \,\} = \infty$. In fact, if $\Omega_{-i}(N)$ is the i^{th} *cosyzygy* of N (since $A^!$ is selfinjective, we can consider cosyzygies), then $\operatorname{lpd}(\Omega_{-i}(N)) > i$. But when A is the polynomial ring $S = k[x_1, \ldots, x_d]$ and $A^!$ is the exterior algebra $E = \bigwedge \langle y_1, \ldots, y_d \rangle$, we have an upper bound of $\operatorname{lpd}(N)$ for $N \in \operatorname{gr} E$ depending only on $\max\{\, \dim_k N_i \mid i \in \mathbb{Z} \,\}$ and d. But before stating this, we recall a result on a upper bound of $\operatorname{reg}(M)$ for $M \in \operatorname{gr} S$.

Theorem 19 (Brodmann and Lashgari, [3])**.** *Let $S == k[x_1, \ldots, x_d]$ be a polynomial ring. Assume that a graded submodule $M \subset S^{\oplus n}$ is generated by elements whose degrees are at most δ. Then we have $\operatorname{reg}(M) \leq n^{d!}(2\delta)^{(d-1)!}$.*

When $n = 1$ (i.e., when M is an ideal), the above bound is given by Bayer and Mumford [1], and sharper than it seems. In fact, for each $m \in \mathbb{N}$, there is an ideal $I \subset k[x_1, \ldots, x_{10m+1}]$ which is generated by elements of degree at most four but satisfies $\operatorname{reg}(I) \geq 2^{2^m} + 1$. For our study on $\operatorname{lpd}(N)$, the case when $\delta = 1$ (but n is general) is essential. When $n = \delta = 1$, we have $\operatorname{reg}(M) = 1$ in the situation of Theorem 19. So I believe that the bound can be largely improved (at least) when $\delta = 1$.

Theorem 20. *Let $E = \bigwedge \langle y_1, \ldots, y_d \rangle$ be an exterior algebra, and $N \in \operatorname{gr} E$. Set $n := \max\{\, \dim_k N_i \mid i \in \mathbb{Z} \,\}$. Then $\operatorname{lpd}(N) \leq n^{d!}2^{(d-1)!}$.*

Proof. Set $L := N' \in \operatorname{gr} E$. (For graded E-modules, we do not have to distinguish left modules form right ones.) By Theorem 18, it suffices to prove $\operatorname{reg}(H^i(\mathcal{F}(L))) + i \leq$

$n^{d!}2^{(d-1)!}$ for each i. We may assume that $i = 0$. Note that $H^0(\mathcal{F}(L))$ is the cohomology of the sequence

$$S \otimes_k L_{-1} \xrightarrow{\partial_{-1}} S \otimes_k L_0 \xrightarrow{\partial_0} S \otimes_k L_1.$$

Since $\mathrm{im}(\partial_0)(-1)$ is a submodule of $S^{\oplus \dim_k L_1}$ generated by elements of degree 1, we have $\mathrm{reg}(\mathrm{im}(\partial_0)) < n^{d!}2^{(d-1)!}$ by Theorem 19. Consider the short exact sequence $0 \to \ker(\partial_0) \to S \otimes_k L_0 \to \mathrm{im}(\partial_0) \to 0$. Since $\mathrm{reg}(S \otimes_k L_0) = 0$, we have $\mathrm{reg}(\ker(\partial_0)) \leq n^{d!}2^{(d-1)!}$. Similarly, we have $\mathrm{reg}(\mathrm{im}(\partial_{-1})) \leq n^{d!}2^{(d-1)!}$ by Theorem 19. By the short exact sequence $0 \to \mathrm{im}(\partial_{-1}) \to \ker(\partial_0) \to H^0(\mathcal{F}(L)) \to 0$, we have $\mathrm{reg}(H^0(\mathcal{F}(L))) \leq n^{d!}2^{(d-1)!}$. $\square$

In a special case, there is much more reasonable bound for $\mathrm{lpd}(N)$.

Definition 21 (Römer, [13]). We say an integer vector $\mathbf{a} = (a_1, \ldots, a_d) \in \mathbb{Z}^d$ is *squarefree*, if $a_i = 0, 1$ for all i. Let $N = \bigoplus_{\mathbf{a} \in \mathbb{Z}^d} N_{\mathbf{a}}$ be a finitely generated $\mathbb{Z}^d$-graded modules over the exterior algebra $E = \bigwedge \langle y_1, \ldots, y_d \rangle$. We say N is *squarefree*, if $N_{\mathbf{a}} \neq 0$ implies that $\mathbf{a}$ is squarefree.

This concept naturally appears in the study of combinatorial commutative algebra (c.f. [14, 16]). For example, all monomial ideals of E are squarefree.

Proposition 22 (Herzog and Römer, [14]). *If N is a squarefree E-module, then we have $\mathrm{lpd}(N) \leq d - 1$.*

In [17], we describe $\mathrm{lpd}(N)$ for a squarefree E-module N in terms of combinatorial commutative algebra. We will show it below in the case when N is a monomial ideal. We also remark that there is a squarefree E-module N with $\mathrm{lpd}(N) = d - 1$.

Set $[d] := \{1, \ldots, d\}$. Let $\Delta \subset 2^{[d]}$ be an (abstract) simplicial complex (i.e., $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$). It is easy to see that $\Delta^\vee := \{\, F \subset [d] \mid [d] \setminus F \not\in \Delta \,\}$ is a simplicial complex again. We also have $\Delta^{\vee\vee} = \Delta$. Set $J_\Delta = (\, \prod_{i \in F} y_i \mid F \subset [d], F \not\in \Delta \,)$ to be a monomial ideal of E. Any monomial ideal of E is given in this way. Similarly, set $I_\Delta = (\, \prod_{i \in F} x_i \mid F \subset [d], F \not\in \Delta \,)$ to be a monomial ideal of S, and call it the *Stanley-Reisner ideal* of Δ. Any squarefree monomial ideal of S is given in this way.

Proposition 23. *For a simplicial complex $\Delta \subset 2^{[d]}$, we have*

$$\mathrm{lpd}(J_\Delta) = \max\{\, i - \mathrm{depth}_S(\, \mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S)\,) \mid 0 \leq i \leq d \,\}. \tag{4.1}$$

Here we set the depth of the 0 module to be $+\infty$.

If $\mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S) \neq 0$, then we have $i - \mathrm{depth}_S(\, \mathrm{Ext}_S^{d-i}(S/I_{\Delta^\vee}, S)\,) \geq 0$. One might think the right side of the equality (4.1) is strange. But the right side of (4.1) equals 0 if and only if $S/I_{\Delta^\vee}$ is *sequentially Cohen-Macaulay* (see [15]). In this sense, $\mathrm{lpd}(J_\Delta)$ measures "how is $S/I_{\Delta^\vee}$ far from sequentially Cohen-Macaulay?".

Corollary 24 (Römer, [13]). *For a simplicial complex $\Delta \subset 2^{[d]}$, the following are equivalent.*

(1) *$J_\Delta \subset E$ is weakly Koszul.*
(2) *$I_\Delta \subset S$ is weakly Koszul.*
(3) *$S/I_{\Delta^\vee}$ is sequentially Cohen-Macaulay.*

We remark that there are many examples of Stanley-Reisner ideals $I_\Delta \subset S$ which are weakly Koszul (dually, Stanley-Reisner rings S/I_Δ which are sequentially Cohen-Macaulay).

Corollary 25. *If $d \geq 3$, then we have* $\mathrm{lpd}(E/J_\Delta) \leq d-2$.

In this moment, I have no idea whether the above bound is (nearly) sharp for large d.

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