A GENERALIZATION OF n-TORSIONFREE MODULES

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Abstract. We consider in this paper two approximation theorems for finitely generated modules over a commutative noetherian ring; one is due to Auslander and Bridger, and the other is due to Auslander and Buchweitz. We shall give a result which implies both of these two theorems.

Key Words: Torsionfree, Semidualizing, Cohen-Macaulay approximation, Contravariantly finite.

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1. Introduction

In the late 1960s, Auslander and Bridger [2] constructed the notion of a certain approximation, which we call in this paper a spherical approximation. This notion says that each of the modules whose nth syzygies are n-torsionfree is described by using an n-spherical module and a module of projective dimension less than n. On the other hand, about two decades later, the notion of a Cohen-Macaulay approximation was introduced and developed by Auslander and Buchweitz [3]. This notion says that over a Cohen-Macaulay local ring with a canonical module, the category of finitely generated modules is obtained by glueing together the subcategory of maximal Cohen-Macaulay modules and the subcategory of modules of finite injective dimension. Cohen-Macaulay approximations have been playing an important role in commutative algebra. In this paper, we set our sight on these two notions. More precisely, we shall consider and generalize the following two theorems.

Theorem 1 (Auslander-Bridger). *The following are equivalent for a finitely generated module M over a commutative noetherian ring R:*

(1) *$\Omega^n M$ is n-torsionfree;*
(2) *There exists an exact sequence $0 \to Y \to X \to M \to 0$ of finitely generated R-modules such that $\mathrm{Ext}_R^i(X, R) = 0$ for $1 \le i \le n$ and $\mathrm{pd}\, Y < n$.*

Theorem 2 (Auslander-Buchweitz). *Let R be a Cohen-Macaulay local ring with a canonical module. Then for every finitely generated R-module M there exists an exact sequence $0 \to Y \to X \to M \to 0$ of finitely generated R-modules such that X is maximal Cohen-Macaulay and $\mathrm{id}\, Y < \infty$.*

The detailed version of this paper has been submitted for publication elsewhere.

2. The existence of n-C-spherical approximations

Throughout the present paper, R is always a commutative noetherian ring, and all R-modules are finitely generated. Auslander and Bridger [2] introduced the notion of an n-torsionfree module.

Definition 3. Let n be an integer. An R-module M is called *n-torsionfree* if $\mathrm{Ext}_R^i(\mathrm{Tr}M, R) = 0$ for $1 \leq i \leq n$.

In this paper, unless otherwise specified, we always denote by n a positive integer, by C an R-module, by $(-)^\dagger$ the C-dual functor $\mathrm{Hom}_R(-, C)$ and by λ_M the natural homomorphism $M \to M^{\dagger\dagger}$ for an R-module M. Note that λ_R can be identified with the homothety map $R \to \mathrm{Hom}_R(C, C)$. We can generalize the notion of an n-torsionfree module as follows.

Definition 4. Let M be an R-module. We say that M is *1-C-torsionfree* if λ_M is a monomorphism. We say that M is *n-C-torsionfree*, where $n \geq 2$, if λ_M is an isomorphism and $\mathrm{Ext}_R^i(M^\dagger, C) = 0$ for all $1 \leq i \leq n-2$.

We denote by $\mathrm{mod}\, R$ the category of finitely generated R-modules. Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}\, R$. An R-homomorphism $f : X \to M$ is called a *right $\mathcal{X}$-approximation* of M if X belongs to $\mathcal{X}$ and the sequence $\mathrm{Hom}_R(-, X) \xrightarrow{(-,f)} \mathrm{Hom}_R(-, M) \to 0$, where $(-, f) = \mathrm{Hom}_R(-, f)$, is exact on $\mathcal{X}$. We say that $\mathcal{X}$ is *contravariantly finite* if any $X \in \mathcal{X}$ has a right $\mathcal{X}$-approximation. For an R-module X, we denote by $\mathrm{add}\, X$ the full subcategory of $\mathrm{mod}\, R$ consisting of all direct summands of finite direct sums of copies of X.

To develop the notion of an n-C-torsionfree module to the utmost extent, we establish the following definition.

Definition 5. We say that C is *1-semidualizing* if λ_R is a monomorphism and $\mathrm{Ext}_R^1(C, C) = 0$. We say that C is *n-semidualizing*, where $n \geq 2$, if λ_R is an isomorphism and $\mathrm{Ext}^i(C, C) = 0$ for all $1 \leq i \leq n$.

The following proposition, which is essentially proved in [6, Proposition 2.5.1], says that there are a lot of n-semidualizing modules.

Proposition 6. *Let R be a Cohen-Macaulay local ring of dimension $d \geq 2$ with an isolated singularity. Let I be an ideal of R which is a maximal Cohen-Macaulay R-module. Then λ_R is an isomorphism and $\mathrm{Ext}_R^i(I, I) = 0$ for every $1 \leq i \leq d-2$. Hence R is d-I-torsionfree, and I is $(d-2)$-semidualizing.*

For an R-module M, we define $C\mathrm{dim}_R\, M$, the $\mathrm{add}\, C$*-resolution dimension* of M, to be the infimum of nonnegative integers n such that there exists an exact sequence $0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to M \to 0$ with each C_i being in $\mathrm{add}\, C$. Note that $\mathrm{add}\, R$-resolution dimension is the same as projective dimension. We make the following definition.

Definition 7. Let M be an R-module.
(1) We say that M is *n-spherical* if $\mathrm{Ext}^i(M, R) = 0$ for all $1 \leq i \leq n$. We call an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules an *n-spherical approximation* if X is n-spherical and $\mathrm{pd}\, Y < n$.

(2) We say that M is *n-C-spherical* if $\mathrm{Ext}^i(M,C)=0$ for all $1 \le i \le n$. We call an exact sequence $0 \to Y \to X \to M \to 0$ of R-modules an *n-C-spherical approximation* if X is n-C-spherical and $C\dim Y < n$.

We notice that an n-C-spherical approximation gives a right approximation:

Proposition 8. *Define two full subcategories of* $\mathrm{mod}\, R$ *as follows:*

$$\mathcal{X} = \{\, X \in \mathrm{mod}\, R \mid X \text{ is } n\text{-}C\text{-spherical} \},$$
$$\mathcal{Y} = \{\, Y \in \mathrm{mod}\, R \mid C\dim Y < n \,\}.$$

Let $0 \to Y \to X \xrightarrow{f} M \to 0$ *be an exact sequence of R-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then the homomorphism f is a right $\mathcal{X}$-approximation of M.*

We give here a lemma.

Lemma 9. *Suppose that* $\mathrm{Ext}^1(C,C) = 0$. *An R-module M is 1-C-torsionfree if and only if there is an exact sequence* $0 \to M \to C_0 \to N \to 0$ *such that* $C_0 \in \mathrm{add}\, C$ *and* $\mathrm{Ext}^1(N,C) = 0$.

Now, we can state and prove the main result of this section.

Theorem 10. *Let C be an n-semidualizing R-module. The following are equivalent for an R-module M:*

(1) *$\Omega^n M$ is n-C-torsionfree;*
(2) *M admits an n-C-spherical approximation.*

Proof. Let $P_\bullet$ be a projective resolution of M.

(1) $\Rightarrow$ (2): We have an exact sequence $0 \to \Omega^{i+1}M \to P_i \to \Omega^i M \to 0$ for each i. Set $X_0 = \Omega^n M$. Note that X_0 is n-C-torsionfree. Lemma 9 implies that there exists an exact sequence $0 \to X_0 \to C_0 \to Z_1 \to 0$ such that $C_0 \in \mathrm{add}\, C$ and $\mathrm{Ext}^1(Z_1, C) = 0$. We make the pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & X_0 & \longrightarrow & C_0 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & P_{n-1} & \longrightarrow & X_1 & \longrightarrow & Z_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega^{n-1}M & = & \Omega^{n-1}M & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Since $\mathrm{Ext}^1(Z_1, C) = 0 = \mathrm{Ext}^1(P_{n-1}, C)$, we have $\mathrm{Ext}^1(X_1, C) = 0$. If $n = 1$, then the middle column is a desired exact sequence.

Let $n \geq 2$. We can easily check that Z_1 is $(n-1)$-C-torsionfree, and that so is X_1. According to Lemma 9, there is an exact sequence $0 \to X_1 \to C_1 \to Z_2 \to 0$ with $C_1 \in \operatorname{add} C$ and $\operatorname{Ext}^1(Z_2, C) = 0$. We make the pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & C_0 \arrow[r, equal] \arrow[d] & C_0 \arrow[d] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^{n-1}M \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Using the bottom row of the above diagram, we make the pushout diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega^{n-1}M \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_{n-2} \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & Z_2 \arrow[r] & 0 \\
 & \Omega^{n-2}M \arrow[r, equal] \arrow[d] & \Omega^{n-2}M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

From the first diagram, we immediately get $C\dim Y_2 < 2$, and $\operatorname{Ext}^2(Z_2, C) = 0$ because $\operatorname{Ext}^1(X_1, C) = 0 = \operatorname{Ext}^2(C_1, C)$. Hence $\operatorname{Ext}^i(Z_2, C) = 0$ for $i = 1, 2$, and we see from the middle row of the second diagram that $\operatorname{Ext}^i(X_2, C) = 0$ for $i = 1, 2$. Thus, if $n = 2$, then the middle column of the second diagram is a desired exact sequence.

Let $n \geq 3$. Then similar arguments to the above claims show that both Z_2 and X_2 are $(n-2)$-C-torsionfree, and Lemma 9 yields an exact sequence $0 \to X_2 \to C_2 \to Z_3 \to 0$

such that $\mathrm{Ext}^1(Z_3, C) = 0$. Similarly to the above, we make two pushout diagrams:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y_2 \arrow[r, equal] \arrow[d] & Y_2 \arrow[d] & & \\
0 \arrow[r] & X_2 \arrow[r] \arrow[d] & C_2 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega^{n-2}M \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & Z_3 \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \Omega^{n-2}M \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[d] & Z_3 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & P_{n-3} \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & Z_3 \arrow[r] & 0 \\
 & \Omega^{n-3}M \arrow[r, equal] \arrow[d] & \Omega^{n-3}M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

If $n = 3$, then the middle column of the second diagram is a desired exact sequence. If $n \geq 4$, then iterating this procedure, we eventually obtain an exact sequence $0 \to Y_n \to X_n \to M \to 0$ such that $\mathrm{Ext}^i(X_n, C) = 0$ for $1 \leq i \leq n$ and $C\dim Y_n < n$.

(2) $\Rightarrow$ (1): Let $0 \to Y \to X \to M \to 0$ be an n-C-spherical approximation of M. Since $C\dim Y < n$, there exists an exact sequence $0 \to C_{n-1} \overset{d_{n-1}}{\to} C_{n-1} \overset{d_{n-2}}{\to} \cdots \overset{d_1}{\to} C_0 \overset{d_0}{\to} Y \to 0$. Put $Y_i = \operatorname{Im} d_i$ for each i. We have exact sequences $0 \to Y_{i+1} \to C_i \to Y_i \to 0$ and

$0 \to \Omega^{i+1}M \to P_i \to \Omega^i M \to 0$ for each i. The following pullback diagram is obtained:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \Omega M \arrow[r, equal] \arrow[d] & \Omega M \arrow[d] & \\
0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y \arrow[r] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The projectivity of P_0 shows that the middle row splits; we have an isomorphism $L \cong Y \oplus P_0$. Adding P_0 to the exact sequence $0 \to Y_1 \to C_0 \to Y \to 0$, we get an exact sequence $0 \to Y_1 \to C_0 \oplus P_0 \to Y \oplus P_0 \to 0$. Thus the following pullback diagram is obtained:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & Y_1 \arrow[r, equal] \arrow[d] & Y_1 \arrow[d] & & \\
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & C_0 \oplus P_0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega M \arrow[r] \arrow[d] & Y \oplus P_0 \arrow[r] \arrow[d] & X \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

Applying a similar argument to the left column of the above diagram, we get exact sequences $0 \to X_{i+1} \to C_i \oplus P_i \to X_i \to 0$ for $0 \le i \le n-1$, where $X_0 = X$ and $X_n = \Omega^n M$. The assumption yields $\mathrm{Ext}^i(X_0, C) = 0 = \mathrm{Ext}^i(C_0 \oplus P_0, C)$ for $1 \le i \le n$, hence we have an exact sequence $0 \to X_0^\dagger \to (C_0 \oplus P_0)^\dagger \to X_1^\dagger \to 0$ and $\mathrm{Ext}^1(X_1, C) = 0$ for $1 \le i \le n-1$. Inductively, for each $0 \le i \le n-1$ an exact sequence $0 \to X_i^\dagger \to (C_i \oplus P_i)^\dagger \to X_{i+1}^\dagger \to 0$ is obtained and $\mathrm{Ext}^j(X_i, C) = 0$ for $1 \le j \le n-i$. We have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d, "\lambda_{X_1}"'] & C_0 \oplus P_0 \arrow[d, "\lambda_{C_0 \oplus P_0}"'] \\
0 \arrow[r] & X_1^{\dagger\dagger} \arrow[r] & (C_0 \oplus P_0)^{\dagger\dagger}
\end{tikzcd}$$

with exact rows. The assumption says that λ_R is injective, and we see that $\lambda_C = \lambda_{R^\dagger}$ is injective. Hence the map $\lambda_{C_0 \oplus P_0}$ is injective, and so is λ_{X_1}. Therefore X_1 is 1-C-torsionfree. If $n \geq 2$, then λ_R is an isomorphism, and so is λ_C. There is a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & X_2 \arrow[r] \arrow[d, "\lambda_{X_2}"'] & C_1 \oplus P_1 \arrow[r] \arrow[d, "\lambda_{C_1 \oplus P_1}"'] & X_1 \arrow[r] \arrow[d, "\lambda_{X_1}"'] & 0 \\
0 \arrow[r] & X_2^{\dagger\dagger} \arrow[r] & (C_1 \oplus P_1)^{\dagger\dagger} \arrow[r] \arrow[d] & X_1^{\dagger\dagger} & \\
 & & 0 & &
\end{tikzcd}$$

with exact rows and columns, and λ_{X_2} is an isomorphism by the snake lemma. Hence X_2 is 2-C-torsionfree. If $n \geq 3$, then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_3 \arrow[r] \arrow[d, "\lambda_{X_3}"'] & C_2 \oplus P_2 \arrow[r] \arrow[d, "\lambda_{C_2 \oplus P_2}"', "\cong"] & X_2 \arrow[r] \arrow[d, "\lambda_{X_2}"', "\cong"] & 0 & \\
0 \arrow[r] & X_3^{\dagger\dagger} \arrow[r] & (C_2 \oplus P_2)^{\dagger\dagger} \arrow[r] & X_2^{\dagger\dagger} \arrow[r] & \mathrm{Ext}^1(X_3^\dagger, C) \arrow[r] & 0
\end{tikzcd}$$

with exact rows. From this diagram it follows that λ_{X_3} is an isomorphism and $\mathrm{Ext}^1(X_3^\dagger, C) = 0$, which means that X_3 is 3-C-torsionfree. Repeating a similar argument, we see that X_i is i-C-torsionfree for every $1 \leq i \leq n$. Therefore $\Omega^n M = X_n$ is n-C-torsionfree, and the proof of the theorem is completed. $\square$

Theorem 1 is a direct corollary of Theorem 10. Theorem 2 is also a corollary of Theorem 10:

Proof of Theorem 2. If $d = 0$, then $0 \to 0 \to M \xrightarrow{=} M \to 0$ is a desired exact sequence. Let $d \geq 1$. Then ω is d-semidualizing, and $\Omega^d M$ is d-ω-torsionfree. Hence Theorem 10 guarantees the existence of an exact sequence $0 \to Y \to X \to M \to 0$ such that X is d-ω-spherical and $\omega\dim Y < d$. Therefore X is maximal Cohen-Macaulay. On the other hand, noting that ω is an indecomposable R-module, we have an exact sequence $0 \to \omega^{l_{d-1}} \to \omega^{l_{d-2}} \to \cdots \to \omega^{l_0} \to Y \to 0$. Decomposing this into short exact sequences and noting that ω has finite injective dimension, one sees that Y also has finite injective dimension. $\square$

3. Modules whose nth syzygies are n-C-torsionfree

We begin with stating the following lemma.

Lemma 11. *Let M be an R-module.*

1. *If R is 1-C-torsionfree, then so is ΩM.*
2. *If R is 2-C-torsionfree, then for each $n \geq 2$ the map $\lambda_{\Omega^n M}$ is a split monomorphism and the cokernel is isomorphic to $\mathrm{Ext}^n(M, C)^\dagger$.*

For R-modules M, N, we define $\operatorname{grade}(M, N)$ by the infimum of integers i such that $\operatorname{Ext}^i(M, N) \neq 0$. One has $\operatorname{grade}(M, N) = \inf\{\operatorname{depth} N_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp} M\}$. We state a criterion for $\Omega^i M$ to be n-C-torsionfree for $1 \leq i \leq n$ in terms of grade, which can be shown by Lemma 11 and induction on n.

Proposition 12. *Let C be an R-module such that R is $(n-1)$-C-torsionfree.*

(1) *If $\Omega^i M$ is i-C-torsionfree for every $1 \leq i \leq n$, then* $\operatorname{grade}(\operatorname{Ext}^i(M, C), C) \geq i - 1$ *for every $1 \leq i \leq n$.*

(2) *The converse also holds if R is n-C-torsionfree.*

Now we want to consider the difference between this condition and the condition that $\Omega^n M$ is n-C-torsionfree.

Lemma 13. *Let C be an R-module such that λ_R is an isomorphism and* $\operatorname{Ext}^i(C, C) = 0$ *for $1 \leq i \leq n$. If M is an R-module with $C\operatorname{dim} M < \infty$, then* $\operatorname{grade}(\operatorname{Ext}^i(M, C), C) \geq i$ *for any $1 \leq i \leq n$.*

Using this lemma, we can show that under the assumption that C is n-semidualizing, $\Omega^i M$ is i-C-torsionfree for $1 \leq i \leq n$ if and only if $\Omega^n M$ is n-C-torsionfree.

Proposition 14. *Let C be an n-semidualizing R-module. The following are equivalent for an R-module M:*

(1) *$\Omega^n M$ is n-C-torsionfree;*

(2) *$\Omega^i M$ is i-C-torsionfree for every $1 \leq i \leq n$.*

Our next aim is to prove the main result of this section. For this, we introduce the following lemma, which will often be used later.

Lemma 15. *Let R be a local ring and r a positive integer. Suppose that λ_R is an isomorphism and* $\operatorname{Ext}^i(C, C) = 0$ *for all $1 \leq i < r$. Then the following hold.*

(1) $\operatorname{depth} R \geq r$ *if and only if* $\operatorname{depth} C \geq r$*.*

(2) *Let R be a Cohen-Macaulay local ring with* $\dim R < r$*. Then C is a maximal Cohen-Macaulay R-module.*

Now we can prove the main result of this section.

Theorem 16. *Suppose that R is n-C-torsionfree. Then the following are equivalent:*

(1) $\operatorname{id}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} < \infty$ *for any* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{depth} R_{\mathfrak{p}} \leq n - 2$*;*

(2) *$\Omega^n M$ is n-C-torsionfree for any R-module M.*

Proof. When $n = 1$, the assertion (1) holds because there is no prime ideal $\mathfrak{p}$ of R satisfying $\operatorname{depth} R_{\mathfrak{p}} \leq n-2$, and the assertion (2) holds by Lemma 11(1). In the following, we consider the case where $n \geq 2$.

(1) $\Rightarrow$ (2): Fix an R-module M. Induction hypothesis shows that $\Omega^i M$ is i-C-torsionfree for $1 \leq i \leq n-1$. By Proposition 12, we have $\operatorname{grade}(\operatorname{Ext}^i(M, C), C) \geq i - 1$ for $1 \leq i \leq n-1$, and it suffices to prove that the inequality $\operatorname{grade}(\operatorname{Ext}^n(M, C), C) \geq n - 1$ holds. Let $\mathfrak{p} \in \operatorname{Spec} R$. If $\operatorname{depth} C_{\mathfrak{p}} \leq n-2$, then $\operatorname{depth} R_{\mathfrak{p}} \leq n - 2$ by Lemma 15(1). The assumption says that $\operatorname{id}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} < \infty$, and $\operatorname{id}_{R_{\mathfrak{p}}} C_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} \leq n - 2$. Therefore $\operatorname{Ext}^n_R(M, C)_{\mathfrak{p}} \cong \operatorname{Ext}^n_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) = 0$. Thus we see that $\operatorname{grade}(\operatorname{Ext}^n(M, C), C) \geq n - 1$, as desired.

(2) $\Rightarrow$ (1): When $n = 2$, Lemma 11(2) implies that $\operatorname{Ext}^2_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) = 0$ for all R-modules M and $\mathfrak{p} \in \operatorname{Ass} C$, because $\operatorname{Ass}(\operatorname{Ext}^2_R(M,C)^\dagger) = \operatorname{Supp}\operatorname{Ext}^2_R(M,C) \cap \operatorname{Ass} C$. The isomorphism $\lambda_R : R \to \operatorname{Hom}(C,C)$ shows that $\operatorname{Ass} C$ coincides with $\operatorname{Ass} R$. Hence, setting $M = \Omega^i_R(R/\mathfrak{p})$, one has $\operatorname{Ext}^{i+2}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), C_\mathfrak{p}) \cong \operatorname{Ext}^2_{R_\mathfrak{p}}((\Omega^i_R(R/\mathfrak{p}))_\mathfrak{p}, C_\mathfrak{p}) = 0$ for any $\mathfrak{p} \in \operatorname{Ass} R$ and any $i > 0$. Therefore $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$ for $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} = 0$.

Let $n \geq 3$. Fix an R-module M. We have an exact sequence $0 \to \Omega^{n+1}M \to P \to \Omega^n M \to 0$ such that P is a projective R-module. From this we get another exact sequence $0 \to (\Omega^n M)^\dagger \to P^\dagger \to (\Omega^{n+1}M)^\dagger \to \operatorname{Ext}^{n+1}(M,C) \to 0$. Decompose this into short exact sequences:

$$\begin{cases} 0 \to (\Omega^n M)^\dagger \to P^\dagger \to N \to 0, \\ 0 \to N \to (\Omega^{n+1}M)^\dagger \to \operatorname{Ext}^{n+1}(M,C) \to 0. \end{cases} \tag{1}$$

Note from the assumption that both $\Omega^n M$ and $\Omega^{n+1}M = \Omega^n(\Omega M)$ are n-C-torsionfree. Since R is n-C-torsionfree, we see from the first sequence in (1) that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{n+1}M \arrow[r] \arrow[d, "\alpha"'] & P \arrow[r] \arrow[d, "\lambda_P"', "\cong"] & \Omega^n M \arrow[r] \arrow[d, "\lambda_{\Omega^n M}"', "\cong"] & 0 & \\
0 \arrow[r] & N^\dagger \arrow[r] & P^{\dagger\dagger} \arrow[r] & (\Omega^n M)^{\dagger\dagger} \arrow[r] & \operatorname{Ext}^1(N,C) \arrow[r] & 0
\end{tikzcd}$$

with exact rows, and $\operatorname{Ext}^i(N,C) = 0$ for $2 \leq i \leq n-2$. This diagram shows that α is an isomorphism and $\operatorname{Ext}^1(N,C) = 0$. The second sequence in (1) gives an exact sequence $0 \to \operatorname{Ext}^{n+1}(M,C)^\dagger \to (\Omega^{n+1}M)^{\dagger\dagger} \xrightarrow{\beta} N^\dagger \to \operatorname{Ext}^1(\operatorname{Ext}^{n+1}(M,C),C) \to 0$ and $\operatorname{Ext}^i(\operatorname{Ext}^{n+1}(M,C),C) = 0$ for $2 \leq i \leq n-2$. Since the diagram

$$\begin{tikzcd}
\Omega^n M \arrow[r, equal] \arrow[d, "\lambda_{\Omega^n M}"', "\cong"] & \Omega^n M \arrow[d, "\alpha"', "\cong"] \\
(\Omega^n M)^{\dagger\dagger} \arrow[r, "\beta"] & N^\dagger
\end{tikzcd}$$

commutes, the map β is an isomorphism, and $\operatorname{Ext}^{n+1}(M,C)^\dagger = 0 = \operatorname{Ext}^1(\operatorname{Ext}^{n+1}(M,C),C)$. Thus we have $\operatorname{Ext}^i(\operatorname{Ext}^{n+1}(M,C),C) = 0$ for every $i \leq n-2$, which means that the inequality $\operatorname{grade}(\operatorname{Ext}^{n+1}(M,C),C) \geq n-1$ holds. Therefore, if $\mathfrak{p}$ is a prime ideal of R with $\operatorname{depth} R_\mathfrak{p} \leq n-2$, then $\operatorname{depth} C_\mathfrak{p} \leq n-2$ by Lemma 15(1), and it follows that $\mathfrak{p}$ does not belong to $\operatorname{Supp}\operatorname{Ext}^{n+1}_R(M,C)$, i.e., $\operatorname{Ext}^{n+1}_{R_\mathfrak{p}}(M_\mathfrak{p}, C_\mathfrak{p}) = 0$. Putting $M = \Omega^i_R(R/\mathfrak{p})$, we obtain $\operatorname{Ext}^{n+1+i}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), C_\mathfrak{p}) \cong \operatorname{Ext}^{n+1}_{R_\mathfrak{p}}((\Omega^i_R(R/\mathfrak{p}))_\mathfrak{p}, C_\mathfrak{p}) = 0$ for any $i > 0$. This implies that $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$, and the proof is completed. $\square$

The lemma below says that over a Gorenstein local ring of dimension $d \geq 2$, any n-semidualizing module is free for $n \geq d$.

Lemma 17. *Let $(R, \mathfrak{m}, k)$ be a d-dimensional Gorenstein local ring. If λ_R is an isomorphism and* $\operatorname{Ext}^i(C,C) = 0$ *for $1 \leq i \leq d$, then $C \cong R$.*

Applying the above lemma, we can get a sufficient condition for R and C to satisfy Theorem 16(1).

Proposition 18. *Suppose that R is n-C-torsionfree and that $R_\mathfrak{p}$ is Gorenstein for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \leq n-2$. Then $\operatorname{id}_{R_\mathfrak{p}} C_\mathfrak{p} < \infty$ for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \leq n-2$. (Hence $\Omega^n M$ is n-C-torsionfree for any R-module M.)*

We have studied the case where the nth syzygies of all R-modules are n-C-torsionfree. As the last result of this paper, we give a result concerning when the nth syzygy of a given module is n-C-torsionfree.

Proposition 19. *Let M be an R-module, and let C be an R-module such that R is n-C-torsionfree. Suppose that $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} < \infty$ for any $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{depth} R_\mathfrak{p} \leq n-2$. Then $\Omega^n M$ is n-C-torsionfree.*

This proposition can be proved by induction on n. Apply Proposition 12, Lemmas 11(1) and 15(1).

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